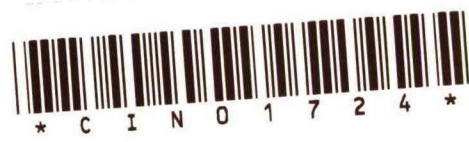


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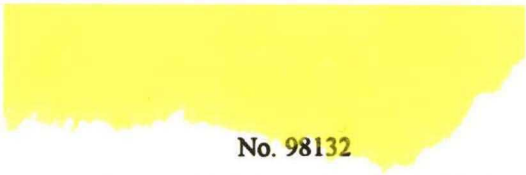


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**TESTING FOR MEAN-VARIANCE SPANNING:  
A SURVEY**

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# Testing for Mean-Variance Spanning: A Survey

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## Abstract

In this paper we present a survey on the various approaches that can be used to test whether the mean-variance frontier of a set of assets spans or intersects the frontier of a larger set of assets. We analyze the restrictions on the return distribution that are needed to have mean-variance spanning or intersection. The paper explores the duality between mean-variance frontiers and volatility bounds, analyzes regression based test procedures for spanning and intersection, and shows how these regression based tests are related to tests for mean-variance efficiency, performance measurement, optimal portfolio choice and specification error bounds. Finally we show how the framework presented in the paper can be used to interpret some well studied empirical issues such as international diversification, currency hedging and multi-factor asset pricing models.

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# 1 Introduction

In recent years the finance literature has witnessed an increasing use of tests for mean-variance spanning and intersection, as introduced by Huberman & Kandel (1987). In this paper we will provide a survey of the literature on testing for mean-variance spanning and intersection, as well as of its relationships with volatility bounds, tests for mean-variance efficiency, performance evaluation and the specification error bounds that have recently been proposed by Hansen & Jagannathan (1997). There exists a vast literature on most of these subjects and the intention here is not to give a complete overview, but merely to illustrate that the concept of mean-variance spanning and intersection provides a framework in which many other results can be understood.

The literature on mean-variance spanning and intersection analyzes the effect that the introduction of additional assets has on the mean-variance frontier. If the mean-variance frontier of the benchmark assets and the frontier of the benchmark plus the new assets have exactly one point in common, this is known as *intersection*. This means that there is one mean-variance utility function for which there is no benefit from adding the new assets. If the mean-variance frontier of the benchmark assets plus the new assets coincides with the frontier of the benchmark assets only, there is *spanning*. In this case no mean-variance investor can benefit from adding the new assets to his (optimal) portfolio of the benchmark assets only. For instance, DeSantis (1995) and Cumby & Glen (1990) consider the question whether US-investors can benefit from international diversification. Taking the viewpoint of a US-investor who initially only invests in the US, these authors study the question whether they can enhance the mean-variance characteristics of their portfolio by also investing in other (developed) markets. Similarly, taking the perspective of a US-investor who invests in the US and (possibly) in other developed markets such as Japan and Europe, DeSantis (1994), Bekaert & Urias (1996), Errunza, Hogan & Hung (1998), and DeRoos, Nijman & Werker (1998a) e.g., investigate whether the investors can improve upon their mean-variance portfolio by investing in emerging markets. As a final example, Glen & Jorion (1993) investigate whether mean-variance investors with a well-diversified international portfolio of stocks and bonds should add currency futures to their portfolio, i.e., whether or not they should hedge the currency risk that arises from their positions in stocks and bonds.

As shown by DeSantis (1994), Ferson, Foerster, & Keim (1993), Ferson



(1995) and Bekaert & Urias (1996), the hypothesis of mean-variance spanning and intersection can be reformulated in terms of the volatility bounds introduced by Hansen & Jagannathan (1991). In that case, the interest is in the question whether a set of additional assets contains information about the volatility of the pricing kernel or the stochastic discount factor that is not already present in the initial set of assets considered by the econometrician. For instance, in the case of emerging markets, the question is whether considering returns from the US-market together with returns from emerging markets produces tighter volatility bounds on the stochastic discount factor than returns from the US-market only.

It turns out that there is a very close link between mean-variance frontiers and volatility bounds for the stochastic discount factors. This duality will be the subject of the next section. The analysis provided in that section will then allow us to study mean-variance spanning and intersection, both in terms of mean-variance frontiers and in terms of volatility bounds. The concept of mean-variance spanning and intersection will formally be introduced in Section 3. In that section it will be also be shown how simple regression techniques can be used to test for mean-variance spanning and intersection. In Section 4 we will consider how conditioning information can be incorporated in the test procedures. In Section 5 we will show how deviations from mean-variance intersection and spanning can be interpreted in terms of performance measures like Jensen's alpha and the Sharpe ratio, and how the regression tests for intersection can be used to derive the new optimal portfolio weights. A brief discussion of the specification error bound introduced by Hansen & Jagannathan (1997) and how this is related to mean-variance intersection will be given in Section 6. As with the performance measures in Section 5, specification error bounds are especially of interest when there is no intersection. Finally, in Section 7 we will illustrate the ideas introduced in Sections 1 through 6 with some applications that have recently received a lot of attention in the literature. This paper will end with a summary.

## **2 Volatility bounds and the duality with mean-variance frontiers**

The purpose of this section is to give an introduction to volatility bounds and mean-variance frontiers and to show the duality between these two fron-

tiers. Because mean-variance spanning and intersection can be defined from volatility bounds as well as from mean-variance frontiers, this section provides a basis for the analysis of mean-variance spanning and intersection in the remainder of the paper.

## 2.1 Volatility bounds

Suppose an investor chooses his portfolio from a set of  $K$  assets, with current prices given by the  $K$ -dimensional vector  $P_t$  and whose payoffs in the next period are given by the vector  $P_{t+1}$  (including dividends and the like). Returns  $R_{t,t+1}$  are payoffs with prices equal to one, i.e.,  $R_{t,t+1} \equiv P_{t,t+1}/P_{t,t}$ . Assuming there are no market frictions such as short sales constraints and transaction costs and assuming that the law of one price holds, there exists a *stochastic discount factor* or *pricing kernel*,  $M_{t+1}$ , such that<sup>1</sup>

$$E[M_{t+1}R_{t+1} | I_t] = \iota_K, \quad (1)$$

where  $\iota_K$  is a  $K$ -dimensional vector containing ones, and  $I_t$  is the information set that is known to the investor at time  $t$ . In the sequel we will use  $E_t[\cdot]$  as shorthand for  $E[\cdot | I_t]$ .

One way to motivate (1) is to look at the discrete time consumption and portfolio problem that an investor solves:

$$\begin{aligned} \max_{\{w_t, C_t\}} E_t\left[\sum_{j=0}^{\infty} \rho^j U(C_{t+j})\right], \\ \text{s.t. } W_{t+1} &= w'_t R_{t+1}(W_t - C_t), \\ w'_t \iota_K &= 1, \quad \forall t \end{aligned} \quad (2)$$

where  $C_t$  is consumption at time  $t$ ,  $W_t$  is the wealth owned by the investor at time  $t$ ,  $\rho$  is the subjective discount factor of the investor, and  $w_t$  is the  $K$ -dimensional vector of portfolio weights that the investor has to choose. The function  $\mathcal{U}(C_t, C_{t+1}, \dots) = \sum_{j=0}^{\infty} \rho^j U(C_{t+j})$  is a strictly increasing and concave time-separable utility function. The first order conditions of problem (2) imply that a valid stochastic discount factor is

$$M_{t+1} = \rho \frac{U'(C_{t+1})}{U'(C_t)} \Big|_{C_t^{opt}, w_t^{opt}},$$

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<sup>1</sup>If, instead of the law of one price, we would impose the stronger condition that there are no arbitrage opportunities, then we would also have that  $M_{t+1} > 0$ .

with  $U'(\cdot)$  being the first derivative of  $U$ . Thus, one way to think about the stochastic discount factor or pricing kernel is as the intertemporal marginal rate of substitution (IMRS).

In many of the problems we consider in this paper, it is convenient to look at a more simple portfolio problem. Usually we will restrict ourselves to one-period portfolio problems, where the agent maximizes his indirect utility of wealth function (see, e.g., Ingersoll (1987), p.66):

$$\begin{aligned} & \max_{\{w\}} E_t[u(W_{t+1})], \\ \text{s.t. } & W_{t+1} = W_t w' R_{t+1}, \\ & w' \iota_K = 1. \end{aligned}$$

In this case a valid stochastic discount factor is  $W_t u'(W_{t+1})/\eta$ , with  $u'(\cdot)$  being the first derivative of the indirect utility function evaluated in the optimal portfolio choice, and  $\eta$  the Lagrange multiplier for the restriction that  $w' \iota_K = 1$ .

The expectation of the stochastic discount factor will be denoted by  $v_t$ , i.e.,  $v_t \equiv E_t[M_{t+1}]$ . The name *stochastic discount factor* refers to the fact that  $M_{t+1}$  discounts payoffs differently in different states of the world. To illustrate this, using the definition of covariance, (1) can be rewritten as

$$\iota_K = E_t[M_{t+1} R_{t+1}] = v_t E_t[R_{t+1}] + \text{Cov}_t[R_{t+1}, M_{t+1}]. \quad (3)$$

The first term in (3) uses  $v_t$  to discount the expected future payoffs, while the second term is a risk adjustment (recall that  $\iota_K$  is the price-vector of the returns  $R_{t+1}$ ). Accordingly, risk premia are determined by the covariance of asset payoffs with  $M_{t+1}$ . If one of the assets is a risk free asset with return  $R_t^f$ , then it follows from the conditional expectation in (1) that  $R_t^f = 1/v_t$ . In the sequel we will usually not impose the presence of such a risk free asset. If a risk free asset is available however, then we can always substitute  $1/R_t^f$  for  $v_t$ .

Equation (1) is the starting point for most asset pricing models. In fact, differences in asset pricing models can be interpreted as differences in the function that each model assigns to  $M_{t+1}$  (see, e.g., Cochrane (1997)). Since each valid stochastic discount factor has to satisfy (1), observed asset returns can be used to derive information about these discount factors. For instance, following Hansen & Jagannathan (1991) it is possible to derive a lower bound on the variance of  $M_{t+1}$ , that each valid stochastic discount factor has to

satisfy, which is known as the *volatility bound*. In this paper, the expectation of the stochastic discount factor will usually be a free parameter. We will denote all discount factors that satisfy (1) and that have expectation  $v$  with  $M(v)_{t+1}$ , and derive a lower bound on the variance of each  $M(v)_{t+1}$ .

Let the expectation and covariance matrix of the returns  $R_{t+1}$  be given by  $\mu_R$  and  $\Sigma_{RR}$  respectively, and assume that all returns are independently and identically distributed (*i.i.d.*), so that the expectations and covariances do not vary over time. This assumption will be relaxed in Section 4 of this paper. Given the set of asset returns  $R_{t+1}$ , let  $m_R(v)_{t+1}$  be a candidate stochastic discount factor that has expectation  $v$  and that is linear in the asset returns:

$$m_R(v)_{t+1} = v + \alpha(v)'(R_{t+1} - \mu_R), \quad (4)$$

where we write  $\alpha(v)$  to indicate that these coefficients are a function of the expectation of  $M(v)_{t+1}$ . Substituting (4) into (1) gives for  $\alpha(v)$ :

$$\alpha(v) = \Sigma_{RR}^{-1}(\iota_K - v\mu_R). \quad (5)$$

Since both  $M(v)_{t+1}$  and  $m_R(v)_{t+1}$  satisfy (1) we have that  $E[(M(v)_{t+1} - m_R(v)_{t+1})R_{t+1}] = 0$ , so the difference between any  $M(v)_{t+1}$  that satisfies (1) and  $m_R(v)_{t+1}$  is orthogonal to  $R_{t+1}$  and therefore to  $m_R(v)_{t+1}$  itself. This implies for the variance of  $M(v)_{t+1}$  that:

$$\begin{aligned} \text{Var}[M(v)_{t+1}] &= \text{Var}[m_R(v)_{t+1}] + \text{Var}[(M(v)_{t+1} - m_R(v)_{t+1})] \\ &\geq \text{Var}[m_R(v)_{t+1}], \end{aligned} \quad (6)$$

which shows that  $m_R(v)_{t+1}$  has the lowest variance of all valid stochastic discount factors  $M(v)_{t+1}$ . This minimum variance can be obtained by combining (4) and (5):

$$\text{Var}[m_R(v)_{t+1}] = (\iota_K - v\mu_R)' \Sigma_{RR}^{-1} (\iota_K - v\mu_R). \quad (7)$$

Thus, any pricing model that aims to price the assets  $R_{t+1}$  correctly, has to yield a pricing kernel that, for a given  $v$ , has a variance at least as large as (7). Equivalently, if we know that agents choose their optimal portfolio from the assets that are in  $R_{t+1}$ , then (7) gives the minimum amount of variation of their IMRS that is needed to be consistent with the distribution of asset returns. Luttmer (1996) extends this kind of analysis taking into account market frictions such as short sales constraints and transaction costs. For the frictionless markets setting, Snow (1991) provides a similar analysis to



derive bounds on other moments of the discount factor as well. Balduzzi & Kallal (1997) show how additional knowledge about risk premia may lead to sharper bounds on the volatility of the discount factor.

## 2.2 Duality between volatility bounds and mean-variance frontiers

So far we have focussed on some of the implications of Equation (1) and the distribution of asset returns for any asset pricing model or utility function, i.e., for any choice of the stochastic discount factor  $M(v)_{t+1}$ . Specifically, we derived the minimum amount of variation in stochastic discount factors that is needed to be consistent with the distribution of asset returns. In this section we will show that there is a close correspondence between these volatility bounds and mean-variance frontiers and that stochastic discount factors that correspond to mean-variance optimizing behavior are the stochastic discount factors with the lowest volatility. Mean-variance optimizing behavior is a special case of the portfolio problem considered before, where the problem the agent faces is  $\max_{\{w\}} E[u(W_{t+1})]$ , and where  $E[u(\cdot)]$  is of the form  $f(w'\mu_R, w'\Sigma_{RR}w)$ , with  $f$  increasing in its first argument and decreasing in its second argument.

For further reference it is useful to define the efficient set variables:

$$A \equiv \iota_K' \Sigma_{RR}^{-1} \iota_K, \quad B \equiv \mu_R' \Sigma_{RR}^{-1} \iota_K, \quad \text{and} \quad C \equiv \mu_R' \Sigma_{RR}^{-1} \mu_R.$$

A mean-variance efficient portfolio  $w^*$  is the solution to the problem

$$\max_{\{w\}} L = w'\mu_R - \gamma w'\Sigma_{RR}w - \eta(w'\iota_K - 1),$$

where  $\gamma$  is the coefficient of risk aversion. From the first order conditions of this problem it follows that a portfolio  $w^*$  is mean-variance efficient if there exist scalars  $\gamma$  and  $\eta$  such that<sup>2</sup>

$$w^* = \gamma^{-1} \Sigma_{RR}^{-1} (\mu_R - \eta \iota_K). \quad (8)$$

Because of the restriction  $w'\iota_K = 1$ , it also follows that  $\gamma = B - A\eta$ , implying that each mean-variance efficient portfolio is uniquely determined when either

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<sup>2</sup>More precisely, these are the minimum variance portfolios, i.e., the portfolios that have minimum variance for a given expected return. The mean-variance efficient portfolios, i.e., the portfolios that also have maximum expected return for a given variance, require in addition that  $\gamma \geq 0$ .

$\gamma$  or  $\eta$  is known, unless  $\eta = B/A$ . It is straightforward to show that for a given mean-variance efficient portfolio  $w^*$ , the Lagrange multiplier  $\eta$  equals the expected return on the zero-beta portfolio of  $w^*$ , i.e., the intercept of the line tangent to the mean-variance frontier at  $w^*$  (in mean-standard deviation space). Since  $B/A$ , is the expected return on the global minimum variance (GMV) portfolio, this is the intercept of the asymptotes of the mean-variance frontier, but there are no lines tangent to the frontier originating at this point (see, e.g., Ingersoll (1987, p.86)).

To show the duality between mean-variance frontiers and volatility bounds, take  $\alpha(v)$  for a given  $v$ , and choose a mean-variance efficient portfolio such that  $\eta = 1/v$ . It follows from (8) and (5) that

$$w^*(v) = \frac{\Sigma_{RR}^{-1}(\mu_R - \frac{1}{v}\iota_K)}{B - \frac{1}{v}A} = \frac{\Sigma_{RR}^{-1}(\iota_K - v\mu_R)}{A - vB} = \frac{\alpha(v)}{\iota_K'\alpha(v)}, \quad (9)$$

which shows that the vector  $\alpha(v)$  is proportional to a mean-variance efficient portfolio with zero-beta return equal to  $1/v$ . Thus, each point on the volatility bound of stochastic discount factors, i.e., each  $\alpha(v)$ , corresponds to a unique point on the mean-variance frontier, i.e., a unique  $w^*(v)$ . The only exception to this result is the case where  $\iota_K'\alpha(v) = 0$ , which is the case if  $v = A/B$ , or equivalently,  $\eta = B/A$ . As already noted, this is the case where the zero-beta return equals the expected return on the global minimum variance portfolio (see also Hansen & Jagannathan (1991)). The duality between the mean-variance frontier of  $R_{t+1}$  and the volatility bound derived from  $R_{t+1}$  can also be seen directly from (5) and (8). Comparing the coefficients  $\alpha(v)$  for the minimum variance stochastic discount factor in (5) and the portfolio weights  $w^*$  in (8) for  $\eta = 1/v$ , it can be seen that the coefficients  $\alpha(v)$  are proportional to the portfolio weights  $w^*$ , where the coefficient of proportionality is equal to  $-\eta/\gamma$ , i.e.,  $w^* = (-\eta/\gamma)\alpha(v)$ . In Appendix A we show graphically which points on the volatility bound correspond to points on the mean-variance frontier.

Summarizing, finding stochastic discount factors that have the lowest variance of all stochastic discount factors that price a set of asset returns  $R_{t+1}$  correctly is tantamount to finding mean-variance efficient portfolios for these same assets  $R_{t+1}$ . In the remainder of this paper we will study the effects of adding new assets to the set of assets available to investors. Although most of the results will be stated in terms of mean-variance frontiers and mean-variance efficient portfolios, it should be kept in mind that there is always a dual interpretation in terms of volatility bounds.

### 3 Mean-variance spanning and intersection

In the previous section we considered the volatility bounds and mean-variance frontiers that can be derived from a given set of  $K$  assets with return vector  $R_{t+1}$ . Suppose now that an investor takes an additional set of  $N$  assets with return vector  $r_{t+1}$  into account in his portfolio problem. The question we are interested in is under what conditions mean-variance efficient portfolios derived from the set of returns  $R_{t+1}$  are also mean-variance efficient for the larger set of  $K + N$  assets  $(R_{t+1}, r_{t+1})$ . This problem was addressed in the seminal paper of Huberman & Kandel (1987). If there is only one value of  $\gamma$  or  $\eta$  for which mean-variance investors can not improve their mean-variance efficient portfolio by including  $r_{t+1}$  in their investment set, the mean-variance frontiers of  $R_{t+1}$  and  $(R_{t+1}, r_{t+1})$  have exactly one point in common, which is referred to as *intersection*. In this case we will say that the mean-variance frontier of  $R_{t+1}$  intersects the mean-variance frontier of  $(R_{t+1}, r_{t+1})$ , or simply that  $R_{t+1}$  intersects  $(R_{t+1}, r_{t+1})$ . If there is no mean-variance investor that can improve his mean-variance efficient portfolio by including  $r_{t+1}$  in his investment set, the mean-variance frontiers of  $R_{t+1}$  and  $(R_{t+1}, r_{t+1})$  coincide, which is referred to as *spanning*. In this case we will say that (the mean-variance frontiers of)  $R_{t+1}$  spans (the mean-variance frontier of)  $(R_{t+1}, r_{t+1})$ .

As suggested by the previous section, and as shown by Ferson, Foerster, & Keim (1993), DeSantis (1994), Ferson (1995) and Bekaert & Urias (1996), the concept of mean-variance spanning and intersection has a dual interpretation in terms of volatility bounds. In terms of volatility bounds mean-variance spanning means that the volatility bound derived from the returns  $R_{t+1}$  is the same as the bound derived from  $(R_{t+1}, r_{t+1})$ . Therefore, the minimum variance stochastic discount factors for  $R_{t+1}$ ,  $m_R(v)_{t+1}$ , are also the minimum variance stochastic discount factors for  $(R_{t+1}, r_{t+1})$ , and the asset returns  $r_{t+1}$  do not provide information about the necessary volatility of stochastic discount factors that is not already present in  $R_{t+1}$ . As will be shown formally below, mean-variance intersection is equivalent to saying that the volatility bounds derived from  $R_{t+1}$  and  $(R_{t+1}, r_{t+1})$  have exactly one point in common. Thus, in case of intersection there is exactly one value of  $v$  for which the minimum variance stochastic discount factor does not change, whereas for all other values of  $v$  it does.

In finite samples it will in general be the case that adding assets causes a shift in the estimated mean-variance frontier and the estimated volatility bound. This shift may very well be the result of estimation error however, and



the main question is whether the observed shift is too large to be attributed to chance. Therefore, to answer the question whether or not the observed shift in the mean-variance frontier is significant in statistical terms, in this section we will also show how regression analysis can be used to test for spanning and intersection.

### 3.1 Spanning and intersection in terms of mean-variance frontiers

To state the problem formally, the hypothesis of mean-variance intersection means that there is a portfolio  $w^*$  which is mean-variance efficient for the smaller set  $R_{t+1}$  and which is also mean-variance efficient for the larger set  $(R_{t+1}, r_{t+1})$ . In the sequel, variables that refer to the smaller set  $R_{t+1}$  ( $r_{t+1}$ ) will be referred to with a subscript  $R$  ( $r$ ), or with their dimension  $K$  ( $N$ ), whereas variables that refer to the larger set  $(R_{t+1}, r_{t+1})$ , will not have any subscript or will have their dimension as subscript,  $K + N$ . Thus,  $w_R$  is a  $K$ -dimensional vector with portfolio weights for the assets in  $R_{t+1}$ , and  $w$  is a  $(K + N)$ -dimensional vector with portfolio weights for all the available assets  $(R_{t+1}, r_{t+1})$ . The hypothesis of mean-variance intersection comes down to the statement that there exists a mean-variance efficient portfolio  $w^*$  of the form

$$w^* = \begin{pmatrix} w_R^* \\ 0_N \end{pmatrix}, \quad (10)$$

i.e., there exist scalars  $\gamma$  and  $\eta$ , such that

$$\mu - \eta \iota_{K+N} = \gamma \Sigma \begin{pmatrix} w_R^* \\ 0_N \end{pmatrix}. \quad (11)$$

If such a portfolio  $w^*$  exists, there is one point on the mean-variance frontier of  $R_{t+1}$  that also lies on the mean-variance frontier of  $(R_{t+1}, r_{t+1})$ . Using obvious notation,  $\mu$  consists of two subvectors  $\mu_R$  and  $\mu_r$ , and  $\Sigma$  consists of submatrices  $\Sigma_{RR}$ ,  $\Sigma_{Rr}$ ,  $\Sigma_{rR}$ , and  $\Sigma_{rr}$ . The first  $K$  rows of (11) imply that

$$\mu_R - \eta \iota_K = \gamma \Sigma_{RR} w_R^* \Leftrightarrow w_R^* = \gamma^{-1} \Sigma_{RR}^{-1} (\mu_R - \eta \iota_K). \quad (12)$$

For one thing, note that (12) simply says that  $w_R^*$  is indeed mean-variance efficient for the smaller set  $R_{t+1}$ .

The next step is to derive the restrictions on the distribution of  $R_{t+1}$  and  $r_{t+1}$  that are equivalent to mean-variance intersection. In order to do so, substitute (12) in the last  $N$  rows of (11) to obtain:

$$\begin{aligned}\mu_r - \eta \iota_N &= \Sigma_{rR} \Sigma_{RR}^{-1} (\mu_R - \eta \iota_K), \Leftrightarrow \\ &(\mu_r - \beta \mu_R) + (\beta \iota_K - \iota_N) \eta = 0,\end{aligned}\quad (13)$$

with  $\beta \equiv \Sigma_{rR} \Sigma_{RR}^{-1}$ . Thus, if there is a portfolio that is mean-variance efficient for the smaller set  $R_{t+1}$  that is also mean-variance efficient for the larger set  $(R_{t+1}, r_{t+1})$ , there must exist a  $\eta$  such that the restriction in (13) holds. It follows immediately from the derivation above that this  $\eta$  is the zero-beta return that corresponds to the portfolio  $w_R^*$  (and  $w^*$ ).

If there is mean-variance spanning then *all* mean-variance efficient portfolios  $w^*$  must be of the form (10), i.e., (11) must be true for *all* values of  $\eta$  and the corresponding  $\gamma$ 's. Going through the same steps, if (11) must hold for any  $\eta$ , (13) must hold for any  $\eta$ , and this can only be the case if

$$\mu_r - \beta \mu_R = 0 \quad \text{and} \quad \beta \iota_K - \iota_N = 0, \quad (14)$$

which are the restrictions imposed by the hypothesis of spanning. If these restrictions on the distribution of  $R_{t+1}$  and  $r_{t+1}$  hold, every point on the mean-variance frontier of  $R_{t+1}$  is also on the mean-variance frontier of  $(R_{t+1}, r_{t+1})$  and the two frontiers coincide.

### 3.2 Spanning and intersection in terms of volatility bounds

In the previous section we defined mean-variance spanning and intersection from the properties of mean-variance efficient portfolios and we derived the equivalent restrictions on the distribution of asset returns. In this section we analyze mean-variance intersection and spanning from the properties of minimum variance stochastic discount factors that price the assets in  $R_{t+1}$  and in  $(R_{t+1}, r_{t+1})$  correctly and we show that this imposes the same restrictions on the distribution of the asset returns. In terms of volatility bounds, the hypothesis of intersection is that there is a value of  $v$  such that the minimum variance stochastic discount factor for  $R_{t+1}$ , i.e.,  $m_R(v)_{t+1}$ , is also the minimum variance stochastic discount factor for the larger set  $(R_{t+1}, r_{t+1})$ . The discount factor  $m_R(v)_{t+1}$  as defined by (4) and (5) is the minimum variance

stochastic discount factor for this larger set if it also prices  $r_{t+1}$  correctly. If  $m_R(v)_{t+1}$  prices both  $R_{t+1}$  and  $r_{t+1}$  correctly, the difference between  $m_R(v)_{t+1}$  and any other  $M(v)_{t+1}$  that prices  $R_{t+1}$  and  $r_{t+1}$  correctly is orthogonal to  $R_{t+1}$  and  $r_{t+1}$ , implying that  $m_R(v)_{t+1}$  must have the lowest variance among all stochastic discount factors  $M(v)_{t+1}$ , by the same reasoning that leads to (6).

Thus, the hypothesis of intersection for volatility bounds can be stated as:

$$\exists v \text{ s.t. } E[r_{t+1}m_R(v)_{t+1}] = \iota_N. \quad (15)$$

To show that this hypothesis imposes the same restrictions on the distribution of  $R_{t+1}$  and  $r_{t+1}$  as in (13), substitute (4) and (5) into (15):

$$\begin{aligned} E[r_{t+1}(v + (R_{t+1} - \mu_R)' \Sigma_{RR}^{-1}(\iota_K - v\mu_R))] &= \iota_N, \Leftrightarrow \\ (\mu_r - \Sigma_{rR} \Sigma_{RR}^{-1} \mu_R)' v + (\Sigma_{rR} \Sigma_{RR}^{-1} \iota_K - \iota_N) &= 0, \Leftrightarrow \\ (\mu_r - \beta \mu_R)' v + (\beta \iota_K - \iota_N) &= 0. \end{aligned} \quad (16)$$

Dividing both sides of (16) by  $v$  shows that the hypothesis of intersection in terms of volatility bounds indeed implies the same restrictions as the hypothesis of intersection in terms of mean-variance frontiers, if we choose  $\eta = 1/v$ . This could be expected beforehand, since from the duality between mean-variance frontiers and volatility bounds in (9) we already knew that the vector  $\alpha_R(v)$  that defines  $m_R(v)_{t+1}$ , is proportional to a mean-variance efficient portfolio with zero-beta return  $\eta = 1/v$ . The hypothesis that  $w^*$  is of the form  $(w_R^* \ 0_N)'$  is therefore equivalent the hypothesis that  $\alpha(v)$  is of the form  $(\alpha_R(v)' \ 0_N)'$ .

By the same logic, the hypothesis of spanning in terms of volatility bounds, requires that  $m_R(v)_{t+1}$  prices the returns  $r_{t+1}$  for *all* values of  $v$ :

$$E[r_{t+1}m_R(v)_{t+1}] = \iota_N, \quad \forall v, \quad (17)$$

since in that case the entire volatility bound derived from  $(R_{t+1}, r_{t+1})$  coincides with the volatility bound derived from  $(R_{t+1})$  only. This requirement implies that (16) holds for all values of  $v$ , and this can only be the case if the restrictions in (14) hold.

### 3.3 Intersection and mean-variance efficiency of a given portfolio

A question that is of obvious interest both from a portfolio choice perspective and from an asset pricing perspective, is the question whether or not a given portfolio  $w^p$  is mean-variance efficient or not. From a portfolio choice perspective, an investor will be interested in whether or not his portfolio has the desired properties of a mean-variance efficient portfolio. From an asset pricing perspective, the frequently analyzed question is, e.g., whether or not the market portfolio is mean-variance efficient as the CAPM predicts.

Denote the return on some portfolio  $w^p$  by  $R_{t+1}^p$  and its expectation by  $\mu^p$ . The question whether or not  $w^p$  is mean-variance efficient with respect to the  $N + 1$  assets  $(R_{t+1}^p, r_{t+1})$ , is obviously a special case of the question whether or not there is mean-variance intersection with  $K = 1$  and  $R_{t+1} = R_{t+1}^p$ , since intersection in this case simply means that the portfolio  $w^p$  is on the mean-variance frontier of  $(R_{t+1}^p, r_{t+1})$ . Therefore, if  $w^p$  is mean-variance efficient for the set  $(R_{t+1}^p, r_{t+1})$ , the following restrictions on the distribution of  $R_{t+1}^p$  and  $r_{t+1}$  should hold:

$$\mu_r = \eta \iota_N + \beta^p (\mu^p - \eta), \quad (18)$$

where  $\beta^p$  is the  $N$ -dimensional vector  $\text{Cov}[r_{t+1}, R_{t+1}^p] / \text{Var}[R_{t+1}^p]$ , and  $\mu^p = E[R_{t+1}^p]$ . When testing for mean-variance efficiency,  $R_{t+1}^p$  is usually the return on a portfolio of  $r_{t+1}$ .

What we want to establish in this section however, is that the hypothesis that the mean-variance frontier of  $R_{t+1}$  ( $K \geq 1$ ) intersects the frontier of  $(R_{t+1}, r_{t+1})$  at a given value of  $\eta = 1/v$ , is tantamount to the hypothesis that the portfolio  $w_R^*$  that is mean-variance efficient for  $R_{t+1}$  and that has  $\eta$  as its zero-beta rate is also mean-variance efficient with respect to  $(R_{t+1}, r_{t+1})$ . Denote the return on  $w_R^*$  as  $R_{t+1}^*$  and its expectation as  $\mu^*$ . Recall that the portfolio  $w_R^*$  is given by the first  $K$  rows of (11)

$$w_R^* = \gamma^{-1} \Sigma_{RR}^{-1} (\mu_R - \eta \iota_K),$$

from which

$$w_R^{*'} (\mu_R - \eta \iota_K) = \gamma w_R^{*'} \Sigma_{RR} w_R^* \Leftrightarrow \gamma = \frac{\mu^* - \eta}{\text{Var}[R_{t+1}^*]}.$$

Substituting these relations into (11) and defining  $\beta^* \equiv \text{Cov}[r_{t+1}, R_{t+1}^*] / \text{Var}[R_{t+1}^*]$ , results in

$$0 = (\mu_r - \eta \iota_N) - \Sigma_{rR} \Sigma_{RR}^{-1} (\mu_R - \eta \iota_K) = (\mu_r - \eta \iota_N) - \gamma \Sigma_{rR} w_R^* =$$

$$(\mu_r - \beta^* \mu) + (\beta^* - \iota_N) \eta. \quad (19)$$

These are the same restrictions as (18) for  $w^p = w^*$ . Thus, the hypothesis of intersection indeed implies the same restrictions on the distribution of  $R_{t+1}$  and  $r_{t+1}$  as the hypothesis that  $w_R^*$  is mean-variance efficient with respect to  $r_{t+1}$ .

### 3.4 Testing for spanning and intersection

So far we derived the restrictions implied by the hypotheses of mean-variance intersection and spanning for the distribution of  $R_{t+1}$  and  $r_{t+1}$ . Huberman & Kandel (1987) showed how regression can be used to test these hypotheses. To see how regression can be used to test for intersection, start from (13):

$$\mu_r - \eta \iota_N = \beta(\mu_R - \eta \iota_K).$$

Replacing the expected returns  $\mu_r$  and  $\mu_R$  with realized returns  $r_{t+1}$  and  $R_{t+1}$ , gives the regression

$$r_{t+1} = \alpha + \beta R_{t+1} + \varepsilon_{t+1}, \quad (20)$$

with  $\alpha = \mu_r - \beta \mu_R$ ,  $\varepsilon_{t+1} = u_{r,t+1} - \beta u_{R,t+1}$ ,  $u_{r,t+1} \equiv r_{t+1} - \mu_r$  and  $u_{R,t+1} \equiv R_{t+1} - \mu_R$ . It can readily be checked that under the null hypotheses of spanning and intersection  $Cov[\varepsilon_{t+1}, R_{t+1}] = 0$ . Notice that  $\alpha$  is an  $N$ -dimensional vector of intercepts,  $\beta$  is a  $N \times K$ -dimensional matrix of slope coefficients, and  $\varepsilon_{t+1}$  is a  $N$ -dimensional vector of error terms. The restrictions imposed by the hypothesis of intersection in (13) can now be stated as

$$\alpha - \eta(\iota_N - \beta \iota_K) = 0. \quad (21)$$

With intersection there are two cases of interest. First, we may be interested in testing for intersection for a given value of the zero-beta rate  $\eta$ . In that case the restrictions in (21) should hold for this specific value of  $\eta$ , which is a set of linear restrictions. In the sequel we will mainly be interested in this case. Second, the interest may be in the question whether there is intersection at some unknown point of the frontier, i.e., for some unknown value of  $\eta$ . In that case the hypothesis is that there exists some  $\eta$  such that the restrictions in (21) hold. This hypothesis can be stated as

$$\alpha_i / (1 - \beta_i \iota_K) = \alpha_j / (1 - \beta_j \iota_K), \quad i, j = 1, \dots, N,$$



where  $\beta_i$  is the  $i$ th row of  $\beta$ . Thus, the hypothesis that there is intersection at some point of the frontier imposes a set of nonlinear restrictions on the regression parameters in (20). Notice that given estimates of  $\alpha_i$  and  $\beta_i$  an estimate of the zero-beta rate for which there is intersection can be obtained from  $\alpha_i/(1 - \beta_i\iota_K)$ . Also note, that testing whether there is intersection at some unknown point of the frontier only makes sense if  $N \geq 2$ , since there is always intersection if  $N = 1$ . (Because there is always one efficient portfolio for which the weight in the new asset is zero.)

Recall that the hypothesis of spanning implies that (21) holds for all values of  $\eta$ . Therefore, going through the same steps, the restrictions imposed by the hypothesis of spanning can be stated as

$$\alpha = 0 \text{ and } \beta\iota_K - \iota_N = 0. \quad (22)$$

The restrictions in terms of the regression model in (20) are intuitively very clear. For instance, the spanning restrictions in (22) state that if there is spanning, then each return of the additional assets,  $r_{i,t+1}$ ,  $i = 1, 2, \dots, N$ , can be written as the return of a portfolio of the benchmark assets  $\beta_i R_{t+1}$ ,  $\beta_i\iota_K = 1$ , plus an error term  $\varepsilon_{i,t+1}$  which has expectation zero and which is orthogonal to the returns  $R_{t+1}$ . Since such an asset can only add to the variance of portfolios of  $R_{t+1}$ , and not to the expected return, mean-variance optimizing agents will not include such an asset in their portfolio. A similar interpretation holds for the intersection restrictions.

If the returns series  $R_{t+1}$  and  $r_{t+1}$  are stationary and ergodic, consistent estimates of the parameters  $\alpha$  and  $\beta$  in (20) are easily obtained using OLS. In writing down the test statistics for (21) and (22), it is convenient to use a different specification of (20), in which all the coefficients  $\alpha$  and  $\beta$  are stacked into one big vector:

$$r_{t+1} = \left( I_N \otimes \begin{pmatrix} 1 & R'_{t+1} \end{pmatrix} \right) b + \varepsilon_{t+1}, \quad (23)$$

where  $b = \text{vec} \left( \begin{pmatrix} \alpha & \beta \end{pmatrix}' \right)$ , a  $(K+1)N$ -dimensional vector. If  $\hat{b}$  is the OLS estimate of  $b$  and  $\hat{Q}$  is a consistent estimate of the asymptotic covariance matrix of  $\hat{b}$ , the hypotheses of intersection and spanning can be tested using a standard Wald test. Defining

$$H(\eta)_{int} \equiv I_N \otimes \begin{pmatrix} 1 & \eta'\iota'_K \end{pmatrix} \text{ and} \quad (24a)$$

$$h(\eta)_{int} \equiv H(\eta)_{int}\hat{b} - \eta\iota_N, \quad (24b)$$

the Wald test-statistic for intersection can be written as

$$\xi_W^{int} = h(\eta)'_{int} \left( H(\eta)_{int} \hat{Q} H(\eta)'_{int} \right)^{-1} h(\eta)_{int}. \quad (25)$$

Similarly, defining

$$H_{span} \equiv I_N \otimes \begin{pmatrix} 1 & 0'_K \\ 0 & \iota'_K \end{pmatrix} \quad \text{and} \quad (26a)$$

$$h_{span} \equiv H_{span} \hat{\delta} - \iota_N \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (26b)$$

the Wald test-statistic for spanning can be written as

$$\xi_W^{span} = h'_{span} \left( H_{span} \hat{Q} H'_{span} \right)^{-1} h_{span}. \quad (27)$$

Under the null hypotheses and standard regularity conditions, the limit distribution of  $\xi_W^{int}$  will be  $\chi^2_N$  and the limit distribution of  $\xi_W^{span}$  will be  $\chi^2_{2N}$ . The test statistics in (25) and (27) have interesting economic interpretations in terms of performance measures. The relationship between tests for intersection and spanning and performance evaluation will be discussed in detail in Section 5.3.

Chen & Knez (1996) and Hall & Knez (1995) propose a test for intersection that is based on (15). Define the deviation from the equality in (15) to be  $\lambda(v)$ :

$$\lambda(v) \equiv E[m_R(v)_{t+1} r_{t+1}] - \iota_N. \quad (28)$$

In Section 5.1 we will interpret  $\lambda(v)$  scaled by  $v$  as a generalization of the well-known Jensen measure. Given an estimate of the parameters  $\alpha_R(v)$  using the sample equivalent of (5):

$$\hat{\alpha}_R(v) = \left( \frac{1}{T} \sum_{t=1}^T (R_t - \bar{R})(R_t - \bar{R})' \right)^{-1} (\iota_K - v \bar{R}),$$

with  $\bar{R}$  the sample mean of  $R_t$ , define  $\hat{\lambda}(v)_t$  as

$$\hat{\lambda}(v)_t \equiv r_t(v + \hat{\alpha}_R(v)'(R_t - \bar{R})) - \iota_N.$$

A test for the hypothesis of intersection,  $\lambda(v) = 0$ , can now be based on

$$\xi_{CK}^{int} = \left( \frac{1}{T} \sum_{t=1}^T \hat{\lambda}(v)_t \right)' \left( \widehat{Var}[\hat{\lambda}(v)_t] \right)^{-1} \left( \frac{1}{T} \sum_{t=1}^T \hat{\lambda}(v)_t \right), \quad (29)$$



where the estimate  $\widehat{Var}[\widehat{\lambda}(v)_t]$  can for instance be obtained using the method suggested by Newey & West (1987). The limit distribution of the test-statistic  $\xi_{CK}^{int}$  is also  $\chi_N^2$ . It is straightforward to show that for  $\eta = 1/v$ ,  $(\frac{1}{T} \sum_{t=1}^T \widehat{\lambda}(v)_t) / v = H(\eta)_{int} \widehat{\delta} - h(\eta)_{int}$ , and that the only difference in the Wald test-statistic in (25) and the statistic proposed in (29) is the way in which the covariance matrix is estimated.

A disadvantage of the test originally proposed by Chen & Knez (1996) is that they test for intersection for a very specific stochastic discount factor, which corresponds to the minimum second moment portfolio. This discount factor can be found by projecting the kernel  $M_{t+1}$  on the asset returns only, excluding the constant. The corresponding portfolio on the mean-variance frontier is the one with the minimum second moment among all portfolios on the frontier, and can graphically be found as the tangency point between the mean-variance frontier and a circle with its centre at the origin. The problem with this portfolio is that it is located at the inefficient part of the frontier, implying that the test used by Chen & Knez (1995) is for intersection at an inefficient portfolio. Therefore it is economically not very interesting, unless there exists a risk free asset. Since in the test statistic in (29) the discount factor  $m_R(v)_{t+1}$  results from a projection of  $M_{t+1}$  on  $R_{t+1}$  plus a constant, this test allows us to test for intersection at any mean-variance efficient portfolio, so this test does not suffer from the problem of the test originally suggested by Chen & Knez.

Alternative tests for the hypotheses of intersection and spanning are suggested by Huberman & Kandel (1987), who propose a likelihood ratio test, and by Snow (1991) and DeSantis (1995), who propose a Generalized Method of Moments (GMM) procedure. This latter procedure is also identical to the *region subset test* suggested by Hansen, Heaton & Luttmer (1995) which is equivalent to a test for intersection. A comparison of the small sample properties of various test-procedures can be found in Bekaert & Urias (1996). The GMM-based test or region subset test is based on the observation that under the null hypotheses of spanning or intersection, the kernel that prices  $R_{t+1}$  and  $r_{t+1}$  correctly is of the form

$$\begin{aligned} m(v)_{t+1} &= v + \alpha_R(v)'(R_{t+1} - \mu_R) + \alpha_r(v)'(r_{t+1} - \mu_r), \\ \text{with } \alpha_r(v) &= 0. \end{aligned}$$

Given that  $\alpha_r(v) = 0$ , a GMM-estimate of the  $K$  parameters in  $\alpha_R(v)$  can

be obtained by using the  $K + N$  sample moments

$$g_T(a_R(v)) = \frac{1}{T} \sum_{t=1}^T \left\{ \begin{pmatrix} R_t \\ r_t \end{pmatrix} (v + a_R(v))' (R_t - \bar{R}) \right\} - \iota_{K+N} = \frac{1}{T} \sum_{t=1}^T g_t(a_R(v)).$$

A consistent estimate of  $\alpha_R(v)$  can therefore be obtained by solving

$$\min_{a_R(v)} g_T(a_R(v))' W_T g_T(a_R(v)) = J_T(a_R(v)), \quad (30)$$

where  $W_T$  is a symmetric nonsingular weighting matrix. Notice that the GMM-estimate of the  $K$  parameters  $\alpha_R(v)$  obtained from (30) is based on  $K + N$  moment restrictions. The  $N$  overidentifying restrictions are derived from the hypothesis that  $m_R(v)_{t+1}$  must also price the  $N$  additional assets  $r_{t+1}$ . Intersection for a given value of  $v$  can now be tested by using the fact that under the null-hypothesis and regularity conditions  $TJ_T(a_R(v))$  is asymptotically  $\chi_N^2$ -distributed. Since spanning implies that (15) holds for (at least) two different values of  $v$ , the GMM-based test can easily be extended by estimating two vectors  $\alpha_R(v_1)$  and  $\alpha_R(v_2)$  simultaneously ( $v_1 \neq v_2$ ) using (30). In this case there are  $2K$  parameters to be estimated with  $2(K + N)$  moment conditions. The test for spanning is therefore a test for the  $2N$  overidentifying restrictions and will asymptotically be  $\chi_{2N}^2$ -distributed under the null-hypothesis of spanning.

## 4 Testing for spanning and intersection with conditioning information

The purpose of this section is to incorporate conditioning information in tests for intersection and spanning. Until now we assumed that returns are independently and identically distributed (*i.i.d.*). However, there is ample evidence that asset returns are to some extent predictable. For instance, stock and bond returns can be predicted from variables like lagged returns, dividend yields, short term interest rates, and default premiums (see, e.g., Ferson (1995)) and futures returns can be predicted from hedging pressure variables (see e.g. DeRoos, Nijman & Veld (1998)) as well as from the spread between spot and forward prices. In Section 4.1 we will show how conditional information can be used in a straightforward way by using *scaled returns* (see, e.g., Cochrane (1997) and Bekaert & Urias (1996)). Although

this is a fairly general and intuitive way of incorporating conditional information, a disadvantage of this method is that the dimension of the estimation and testing problem - increases quickly. In Section 4.2 we show that this problem can be circumvented if it is assumed that variances and covariances are constant, while expected returns are allowed to vary over time. Under this assumption it is shown that the conditioning variables can easily be accounted for by using them as additional regressors. The restrictions for the intersection and spanning hypotheses appear to be very similar to the restrictions in case returns are independently and identically distributed. This way of incorporating conditional variables also has the additional advantage that the regression estimates indicate under what economic circumstances, i.e., for what values of the conditioning variables, intersection and spanning can or can not be rejected. Finally, in Section 4.3 we will discuss the use of conditioning variables as, e.g., in Shanken (1990) and Ferson & Schadt (1996). In that case variances and covariances are allowed to vary over time as well.

#### 4.1 Incorporating conditional information using scaled returns

Suppose that  $z_t$  is a  $(L - 1)$ -dimensional vector of instruments that has predictive power for  $R_{t+1}$  and  $r_{t+1}$ , and define the  $L$ -dimensional vector  $Z_t$  as  $Z_t \equiv (1 \ z_t)'$ . A common way to use these instruments is to look at *scaled returns*:  $Z_t \otimes R_{t+1}$ . If  $M_{t+1}$  is a valid stochastic discount factor, then from (1) we have:

$$E[M_{t+1}(Z_t \otimes R_{t+1}) \mid I_t] = Z_t \otimes \iota_K.$$

Taking unconditional expectations, this yields

$$E[M_{t+1}(Z_t \otimes R_{t+1})] = E[Z_t \otimes \iota_K]. \quad (31)$$

Thus, the scaled return  $Z_{i,t}R_{j,t+1}$  has an average price equal to  $E[Z_{i,t}]$ . The scaled returns can be interpreted as the payoffs of a strategy where each period an amount equal to  $Z_{i,t}$  dollars is invested in a security, yielding a payoff equal to  $Z_{i,t}R_{j,t+1}$ . Therefore, we can also think of  $Z_t \otimes R_{t+1}$  as the returns on *managed portfolios* (see, e.g., Cochrane (1997)). By allowing for such managed portfolios, we take into account that investors may use dynamic strategies, based on the realized values of  $Z_t$ . In effect this increases the set of available assets by a factor  $L$ .

To simplify notation, denote the  $(L \times K)$ -dimensional vector  $Z_t \otimes R_{t+1}$  by  $R_{t+1}^Z$ . Also, denote the  $(L \times K)$ -dimensional vector  $E[Z_t \otimes \iota_K]$  by  $q_K$ . For further reference,  $r_{t+1}^Z$  and  $q_N$  are defined in a completely analogous way. Valid stochastic discount factors  $M_{t+1}^Z$  now have to satisfy

$$E[M_{t+1}^Z R_{t+1}^Z] = q_K. \quad (32)$$

Following the same line of reasoning as in Sections 2.1 and 2.2, it is straightforward to show that the minimum variance stochastic discount factor with expectation  $v$  is given by

$$\begin{aligned} m_{R,t+1}^Z &= v + \alpha^Z(v)'(R_{t+1}^Z - \mu_R^Z), \\ \alpha^Z(v) &= (\Sigma_{RR}^Z)^{-1}(q_K - v\mu_R^Z). \end{aligned} \quad (33)$$

This expression for the volatility bound is a straightforward generalization of the one given in (4) and (5). The restrictions imposed by the hypotheses of intersection and spanning also turn out to be very similar to the ones given in previous sections, as we will see below.

Thus, conditioning information can be incorporated by including managed portfolios, the returns of which depend on the conditioning variables. If there is to be conditional intersection or spanning of  $r_{t+1}$  by  $R_{t+1}$ , the unconditional volatility bound (or mean-variance frontier) of  $R_{t+1}^Z$  must intersect or span the volatility bound (or mean-variance frontier) of  $(R_{t+1}^Z, r_{t+1}^Z)$ . The interest is therefore in the returns  $R_{t+1}$  and  $r_{t+1}$  themselves *plus* the returns on all the managed portfolios. Intersection or spanning is equivalent to

$$E[r_{t+1}^Z m_{R,t+1}^Z(v)] = q_N, \quad (34)$$

for one value of  $v$  or for all values of  $v$  respectively. To see which restrictions these hypotheses imply, substitute (33) into (34) to obtain

$$(\mu_r^Z - \beta^Z \mu_R^Z)v + (\beta^Z q_K - q_N) = 0, \quad (35)$$

for intersection, and

$$(\mu_r^Z - \beta^Z \mu_R^Z) = 0, \quad \text{and} \quad (\beta^Z q_K - q_N) = 0, \quad (36)$$

for spanning. Here  $\beta^Z$  is a  $(L \times N) \times (L \times K)$  matrix with slope coefficients from a regression of  $r_{t+1}^Z$  on  $R_{t+1}^Z$  plus a constant. These restrictions are also given in Bekaert & Urias (1996).



The similarity with the case in which there was no conditioning information is obvious. The only difference in the restrictions is that in (35) and (36) we have  $(\beta^Z q_K - q_N)$  instead of  $(\beta \iota_K - \iota_N)$ . The fact that  $q_K$  and  $q_N$  enter the restrictions reflects the fact that  $R_{t+1}^Z$  and  $r_{t+1}^Z$  are not really returns, in the sense that their current prices are not necessarily equal to one. The average prices of  $R_{t+1}^Z$  and  $r_{t+1}^Z$  are instead given by  $q_K$  and  $q_N$ . The average cost of the managed portfolios with payoff vector  $r_{t+1}^Z$  is given by the vector  $q_N$ , and the cost of the mimicking portfolios from  $R_{t+1}^Z$  is given by  $\beta^Z q_K$ . The interpretation of the restrictions given in Section 3.4 is therefore still valid.

The main disadvantage of this way of incorporating conditioning information is that the number of parameters to be estimated as well as the number of restrictions to be tested grows rapidly with the number of instruments  $L$ . The number of exogenous variables equals  $K \times L$  and the number of restrictions to be tested equals  $N \times L$  for the hypothesis of intersection, and  $2N \times L$  for the hypothesis of spanning. This is the case because for each new instrument there are  $K$  new managed portfolios to be considered for the assets in  $R_{t+1}$  and  $N$  additional managed portfolios for the assets in  $r_{t+1}$ .

This problem can at least partially be circumvented if we are willing to assume a more specific form of predictability. Specifically, in the next section we make the assumption that only the expected returns of  $R_{t+1}$  and  $r_{t+1}$  depend linearly on the instruments  $z_t$ , whereas all variances and covariances are constants. In Section 4.3 the slope coefficients  $\beta$  are assumed to depend linearly on the instruments, which also allows for a straightforward way of incorporating conditional information in the regression framework to test for intersection and spanning.

## 4.2 Expected returns linear in the conditional variables

In this section we assume that there is a specific form of predictability, which allows us to incorporate conditioning information in a straightforward way in the regression framework for spanning and intersection. The assumption made is that expected returns are linear in the conditional variables and that returns are conditional homoskedastic. This way of incorporating conditioning information is used for instance in Campbell & Viceira (1998) and DeRoos, Nijman & Werker (1998b). The assumption we make is that

$$E_t[R_{t+1}] = \gamma'_R Z_t, \quad (37)$$

$$E_t[r_{t+1}] = \gamma'_r Z_t,$$

and the variances and covariances of  $R_{t+1}$  and  $r_{t+1}$  conditional on  $Z_t$  are given by  $Var[R_{t+1} | Z_t] = \Omega_{RR}$ ,  $Var[r_{t+1} | Z_t] = \Omega_{rr}$ , and  $Cov[r_{t+1}, R_{t+1} | Z_t] = \Omega_{rR}$ . Starting from (1), the minimum variance stochastic discount factor is, in this particular setting, given by

$$\begin{aligned} m_R(v)_{t+1} &= v + \alpha(v)'_t (R_{t+1} - E_t[R_{t+1}]), \\ \alpha(v)_t &= \Omega_{RR}^{-1}(\iota_K - v E_t[R_{t+1}]). \end{aligned} \quad (38)$$

If there is intersection,  $m_R(v)_{t+1}$  must price  $r_{t+1}$  correctly conditional on  $Z_t$ , which results in

$$\begin{aligned} \iota_N &= E_t[r_{t+1} m_R(v)_{t+1}] = v \gamma'_r Z_t + \Omega_{rR} \Omega_{RR}^{-1}(\iota_K - \gamma'_R Z_t) \\ \Leftrightarrow (\gamma'_r - \Omega_{rR} \Omega_{RR}^{-1} \gamma'_R) Z_t v + (\Omega_{rR} \Omega_{RR}^{-1} \iota_K - \iota_N) &= 0. \end{aligned} \quad (39)$$

In case there is spanning this condition must again hold for every  $v$ , implying

$$(\gamma'_r - \Omega_{rR} \Omega_{RR}^{-1} \gamma'_R) Z_t = 0 \quad \text{and} \quad (\Omega_{rR} \Omega_{RR}^{-1} \iota_K - \iota_N) = 0. \quad (40)$$

It turns out that the regression framework that we used to test for spanning and intersection can easily be modified to test the restrictions in (39) and (40). In Appendix B it is shown that in the regression

$$r_{t+1} = \gamma Z_t + \delta R_{t+1} + u_{t+1}, \quad (41)$$

with  $E[u_{t+1} Z_t] = 0$ , and  $E[u_{t+1} R_{t+1}] = 0$ , the OLS-estimates of  $\gamma$  and  $\delta$  are consistent estimates of  $(\gamma'_r - \Omega_{rR} \Omega_{RR}^{-1} \gamma'_R)$  and  $(\Omega_{rR} \Omega_{RR}^{-1} \iota_K - \iota_N)$  respectively, which are the parameters of interest in the restrictions in (39) and (40). The hypotheses of intersection and spanning can therefore be based on the OLS-estimates of (41). The hypothesis that there is intersection for a given value of  $v$  and  $Z_t$  can be tested by testing the restrictions

$$\gamma Z_t v + (\delta \iota_K - \iota_N) = 0, \quad (42)$$

and the hypothesis of spanning by testing the restrictions

$$\gamma Z_t = 0 \quad \text{and} \quad (\delta \iota_K - \iota_N) = 0. \quad (43)$$

These restrictions are very similar to the restrictions implied by intersection and spanning in the unconditional case, except that the intercept  $\alpha$  in (20) is replaced by  $\gamma Z_t$ .

It can easily be seen from (42) and (43) that the number of restrictions to be tested for intersection and spanning is the same as in the unconditional case, which makes this method of incorporating conditional information somewhat more attractive than using scaled returns. Note that the hypotheses underlying (42) and (43) are that there is intersection or spanning for a particular value of  $Z_t$ , i.e., for a particular state of the economy. This has the additional advantage that the regression estimates of (41) allow us to make statements about the question in what states of the economy it will be useful to invest in  $r_{t+1}$  as well as in  $R_{t+1}$ . For instance, given the estimates of  $\gamma$  and  $\delta$  in (41) and the concomitant covariance matrix, it is possible to derive confidence intervals for the values of  $Z_t$  for which there can be intersection or spanning.

If the hypothesis of interest is whether there is spanning regardless of the state of the economy, the restrictions in (43) should hold for all values of  $z_t$ , implying that each element of  $\gamma$  should be equal to 0. In that case, with  $L$  instruments and  $N$  assets in  $r_{t+1}$ , there are  $L \times N$  restrictions to be tested, which, although smaller than the  $2 \times L \times N$  restrictions in (36), can be a large number. Also, as follows readily from (42) and (43), in this case the hypothesis of intersection and the hypothesis of spanning both imply the same restrictions.

### 4.3 Slope coefficients $\beta$ linear in the conditional variables

An alternative way of incorporating conditional information in the regression framework is suggested by Shanken (1990) and Ferson & Schadt (1996) e.g., where the slope coefficients  $\beta$  are assumed to be a linear function of the instruments. In the regression in (20),

$$r_{t+1} = \alpha + \beta R_{t+1} + \varepsilon_{t+1},$$

Shanken (1990) simply assumes that

$$\begin{aligned}\alpha &= a_0 + a_1 z_t, \\ \beta &= b_0 + b_1 z_t,\end{aligned}\tag{44}$$

where  $z_t$  are now supposed to be demeaned variables. Ferson & Schadt (1996) motivate (44) as a first order Taylor-series expansion for a general



dependence of  $\beta$  on  $Z_t = (1 \ z_t)'$ . Let  $Cov[r_{t+1}, R_{t+1} \mid Z_t] = \Sigma_{rR}(Z_t)$ , and  $Var[R_{t+1} \mid Z_t] = \Sigma_{RR}(Z_t)$ , where  $\Sigma(\cdot)$  indicates some functional form for the covariance matrix. Starting from (13) intersection for a given zero-beta rate  $\eta = 1/v$  conditional on  $Z_t$  means

$$\begin{aligned} E[r_{t+1} - \eta \iota_N] &= \beta(Z_t) E[R_{t+1} - \eta \iota_K] \Leftrightarrow \\ r_{t+1} - \eta \iota_N &= \beta(Z_t)(R_{t+1} - \eta \iota_K) + u_{t+1}, \end{aligned}$$

with  $\beta(Z_t) = \Sigma_{rR}(Z_t) \Sigma_{RR}(Z_t)^{-1}$ ,  $u_{t+1} \equiv (r_{t+1} - \beta(Z_t)R_{t+1}) - (E[r_{t+1}] - \beta(Z_t)E[R_{t+1}])$ , and  $E[u_{t+1} \mid Z_t] = 0$ . Ferson & Schadt (1996) suggest a linear approximation of  $\beta(Z_t)$ :

$$\beta(Z_t) \approx b_0 + b_1 z_t, \quad (45)$$

from which

$$\begin{aligned} r_{t+1} &= a_0 + a_1 z_t + b_0 R_{t+1} + b_1 z_t R_{t+1} + \varepsilon_{t+1}, \\ a_0 &= \eta(\iota_N - b_0 \iota_K), \\ a_1 &= -\eta b_1 \iota_K, \end{aligned} \quad (46)$$

with  $\varepsilon_{t+1} = u_{t+1} + (\beta(Z_t) - b_0 - b_1 z_t)(R_{t+1} - \eta \iota_K)$ , for which it is assumed that  $E[\varepsilon_{t+1} \mid Z_t] = 0$ . This gives precisely the regression in (20) where the regression parameters are linear in the instruments as assumed by Shanken (1990).

Intersection for a given value of  $\eta = 1/v$  and  $z_t$  can now be tested by testing the restrictions that

$$(a_0 + a_1 z_t) + \{(b_0 + b_1 z_t)\iota_K - \iota_N\}\eta = 0. \quad (47)$$

As in the previous section, these restrictions give the additional advantage that statements can be made as in which state of the economy, i.e., for which values of  $z_t$  there is intersection. If there is intersection for all values of  $z_t$ , this implies

$$\begin{aligned} a_0 + (b_0 \iota_K - \iota_N)\eta &= 0, \\ a_1 + b_1 \iota_K \eta &= 0. \end{aligned}$$

Spanning for a given value of  $z_t$  is equivalent to

$$\begin{aligned} a_0 + a_1 z_t &= 0, \\ (b_0 + b_1 z_t)\iota_K &= \iota_N. \end{aligned} \quad (48)$$

Again, for a specific value of  $z_t$ , i.e., for specific economic conditions, these restrictions can easily be tested in the regression framework outlined above. If there is to be spanning under all economic conditions the restrictions are

$$\begin{aligned} a_0 &= 0, \\ b_{0K} &= \iota_N, \\ a_1 &= 0, \\ b_1 &= 0. \end{aligned}$$

If there are  $L$  instruments (including a constant) with  $K$  benchmark assets and  $N$  new assets, we now have  $(K+1) \times N \times L$  restrictions to test, which is even larger than with the scaled returns in Section 4.1. Also, the numbers of parameters to be estimated is  $(K+1) \times N \times L$ . Thus, in terms of the number of parameters and the number of restrictions, this approach does not offer additional benefits over the use of scaled returns. However, this approach does have the benefit that it shows under what economic circumstances there may or may not be intersection or spanning.

Notice that this way of incorporating conditional information is very similar to the one suggested in the previous section. The restrictions on the regression parameters in (46) are analogous to the ones on the parameters in (41). The main difference arises because the slope coefficients for  $R_{t+1}$  also depend on the instruments, implying that the interaction term  $z_t R_{t+1}$  should also be included in the regression. It is easy to see that the approach in the previous section can be interpreted as a special case of the approach outlined here, where only the intercepts in (20) are a function of the instruments  $z_t$ , whereas the slope coefficients are constant.

Summarizing, we have shown that a number of approaches is available to incorporate conditioning information in tests for intersection and spanning. Using either scaled returns or regression coefficients that are linear functions of the instruments, the regression approach outlined in Section 3 can easily be extended to test for intersection or spanning. The restrictions implied by the hypotheses of intersection and spanning are very similar to the case where there is no conditioning information (i.e., where the only instrument is a constant) and have very similar interpretations as well.

## 5 The relation between spanning tests, performance evaluation and optimal portfolio weights

So far the focus has been on the restrictions that are implied by the hypotheses of intersection and spanning on the distribution of  $R_{t+1}$  and  $r_{t+1}$  and on how these hypotheses can be tested. In this section the interest will be in the deviations from the restrictions. We will show that the test statistics and regression estimates have clear interpretations in terms of performance measures like Jensen's alpha and the Sharpe ratio as well as in terms of the new optimal portfolio weights. Since it is natural to think about these performance measures in terms of mean-variance efficient portfolios, most of the analysis in this section will be in terms of mean-variance frontiers rather than volatility bounds. Nonetheless, the duality between these two frontiers also holds for these performance measures. Interpretations of tests for mean-variance efficiency, intersection, and spanning in terms of performance measures can also be found in Gibbons, Ross & Shanken (1989), Jobson & Korkie (1982, 1984, 1989), and Kandel & Stambaugh (1989).

### 5.1 Performance measures

To set the stage, define the vector of *Jensen's alphas*, or *Jensen performance measures*,  $\alpha_J(\eta)$ , as the intercepts in a regression of the  $N$  excess returns ( $r_{t+1} - \eta\iota_N$ ) on the excess returns of the  $K$  benchmark assets, ( $R_{t+1} - \eta\iota_K$ ):

$$r_{t+1} - \eta\iota_N = \alpha_J(\eta) + \beta(R_{t+1} - \eta\iota_K) + \varepsilon_{t+1}, \quad (49)$$

with  $E[\varepsilon_{t+1}] = E[\varepsilon_{t+1}R_{t+1}] = 0$ . Since it is not assumed that there exists a risk free asset, we define excess returns as the return on an asset or portfolio in excess of a given zero-beta rate  $\eta$ . Alternatively, when regressing  $r_{t+1}$  on  $R_{t+1}$  as in (20), it follows that Jensen's alpha is equal to

$$\alpha_J(\eta) = \alpha + (\beta\iota_K - \iota_N)\eta, \quad (50)$$

where  $\alpha = \mu_r - \beta\mu_R$  and  $\beta = \Sigma_{rR}\Sigma_{RR}^{-1}$ . Notice from this expression that the hypothesis that there is intersection for a given value of  $\eta$  is equivalent to the hypothesis that the Jensen performance measure is zero, i.e.,  $\alpha_J(\eta) = 0$ . Similarly, the hypothesis of spanning is equivalent to the hypothesis that

$\alpha_J(\eta) = 0, \forall \eta$ . Recall from Section 3.3, that the regression in (49) produces the same intercept  $\alpha_J(\eta)$  as a regression of  $r_{t+1} - \eta \iota_N$  on the excess return of a portfolio  $w_R^*$  that is mean-variance efficient for  $R_{t+1}$  and that has  $\eta$  as its zero beta rate, i.e.,

$$r_{t+1} - \eta \iota_N = \alpha_J(\eta) + \beta^*(R_{t+1}^* - \eta) + \varepsilon_{t+1}.$$

It is common in the literature to define Jensen's alpha as the intercept of a regression of  $r_{t+1}$  in excess of the risk free rate on the return of the market portfolio in excess of the risk free rate. The definition in (49) is more general and has this more traditional definition as a special case if there exists a risk free asset ( $\eta = R_t^f$ ) and if the market portfolio is mean-variance efficient ( $R_{t+1}^* = R_{t+1}^m$ ). The Jensen measure in (49) is also referred to as the *generalized* Jensen measure. Given the minimum variance stochastic discount factor  $m_R(v)_{t+1}$  as defined in (4) and (5), it can easily be seen that the generalized Jensen measure is also equal to  $\lambda(v)/v$  as defined in (28).

The *Sharpe ratio* of a portfolio with return  $R_{t+1}^p$  is defined as the expected excess portfolio return, divided by the standard deviation of portfolio return,

$$Sh(R_{t+1}^p, \eta) \equiv \frac{E[R_{t+1}^p] - \eta}{\sigma(R_{t+1}^p)}.$$

By definition, for a given expected portfolio return, or for a given standard deviation of portfolio return, the maximum attainable (absolute) Sharpe ratio is the Sharpe ratio of the minimum-variance efficient portfolio. For a minimum-variance efficient portfolio  $w_R^*$  of the  $K$  assets  $R_{t+1}$  with zero-beta rate  $\eta$ , the Sharpe ratio is equal to the slope of the line tangent to the frontier originating at  $(0, \eta)$  in mean-standard deviation space, and is denoted by  $\theta_R(\eta)$ :

$$\theta_R(\eta) = \frac{E[R_{t+1}^*] - \eta}{\sigma(R_{t+1}^*)}, \quad (51)$$

where  $R_{t+1}^* \equiv w^{*'} R_{t+1}$ .

Although both Jensen's alpha and the Sharpe ratio are used as performance measures, there is an important difference between the two. Whereas the Sharpe ratio is defined in terms of the characteristics of one portfolio (the expected excess portfolio return and its standard deviation), Jensen's alpha is defined in terms of one asset or portfolio relative to another. Sharpe ratios answer the question whether one portfolio is to be preferred over another, whereas Jensen's alpha answers the question whether investors can improve

the efficiency of their portfolio by investing in the asset. However, there is a close relation between the two measures, in that Jensen's alphas together with the covariance matrix of the error terms  $\varepsilon_{t+1}$  in (20) (and (49)) determine the potential improvement in the maximum attainable Sharpe ratio from adding the new assets  $r_{t+1}$ . Recall from Section 2.2 that we defined the variables  $A \equiv \iota' \Sigma^{-1} \iota$ ,  $B \equiv \mu' \Sigma^{-1} \iota$ , and  $C \equiv \mu' \Sigma^{-1} \mu$ . For the set  $R_{t+1}$  these variables will be denoted as  $A_R$ ,  $B_R$ , and  $C_R$ , whereas the absence of subscripts implies that these variables refer to the larger set  $(R_{t+1}, r_{t+1})$ . Using partitioned inverses, notice that

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{RR} & \Sigma_{Rr} \\ \Sigma_{rR} & \Sigma_{rr} \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma_{RR}^{-1} + \beta' \Sigma_{\varepsilon\varepsilon}^{-1} \beta & -\beta' \Sigma_{\varepsilon\varepsilon}^{-1} \\ -\Sigma_{\varepsilon\varepsilon}^{-1} \beta & \Sigma_{\varepsilon\varepsilon}^{-1} \end{pmatrix}. \quad (52)$$

From this, it follows that

$$\begin{aligned} A &= \iota_K' \Sigma_{RR}^{-1} \iota_K + \iota_K' \beta' \Sigma_{\varepsilon\varepsilon}^{-1} \beta \iota_K - 2 \iota_K' \beta' \Sigma_{\varepsilon\varepsilon}^{-1} \iota_N + \iota_N' \Sigma_{\varepsilon\varepsilon}^{-1} \iota_N \\ &= A_R + (\beta \iota_K - \iota_N)' \Sigma_{\varepsilon\varepsilon}^{-1} (\beta \iota_K - \iota_N), \end{aligned} \quad (53)$$

where  $\beta = \Sigma_{rR} \Sigma_{RR}^{-1}$  and  $\Sigma_{\varepsilon\varepsilon}$  is the covariance matrix of  $\varepsilon_{t+1}$ , the error term in the regression in (20). In a similar way it can easily be shown that

$$B = B_R + \alpha' \Sigma_{\varepsilon\varepsilon}^{-1} (\iota_N - \beta \iota_K), \quad (54a)$$

$$C = C_R + \alpha' \Sigma_{\varepsilon\varepsilon}^{-1} \alpha, \quad (54b)$$

where  $\alpha = \mu_r - \beta \mu_R$ , the intercept in the regression in (20).

It is easy to show that for a given  $\eta$ , the Sharpe ratio of a mean-variance efficient portfolio  $w_R^*$  can be written as

$$\theta_R(\eta) = (C_R - 2B_R\eta + A_R\eta^2)^{1/2}. \quad (55)$$

A similar expression holds of course for  $\theta(\eta)$ , the maximum attainable Sharpe ratio of the larger set  $(R_{t+1}, r_{t+1})$ . Combined with (53) and (54) this gives for the squared Sharpe ratio

$$\begin{aligned} \theta(\eta)^2 &= C - 2B\eta + A\eta^2 \\ &= (C_R - 2B_R\eta + A_R\eta^2) \\ &\quad + (\alpha' \Sigma_{\varepsilon\varepsilon}^{-1} \alpha - 2\alpha' \Sigma_{\varepsilon\varepsilon}^{-1} (\iota_N - \beta \iota_K)\eta + (\iota_N - \beta \iota_K)' \Sigma_{\varepsilon\varepsilon}^{-1} (\iota_N - \beta \iota_K)\eta^2) \\ &= \theta_R(\eta)^2 + \alpha_J(\eta)' \Sigma_{\varepsilon\varepsilon}^{-1} \alpha_J(\eta). \end{aligned} \quad (56)$$

Thus, the change in maximum attainable squared Sharpe ratios equals the inner product of the vector of Jensen's alphas,  $\alpha_J(\eta)$ , weighted by the inverse of the covariance matrix of  $\varepsilon_{t+1}$ .<sup>3</sup> If there is only one new asset,  $N = 1$ , the

<sup>3</sup>This result can also be found in Jobson & Korkie (1984) for instance.



term  $\alpha_J(\eta)/\sigma(\varepsilon)$  is known as the *adjusted* Jensen measure or the appraisal ratio. Notice once more that  $\theta_R(\eta)$  and  $\theta(\eta)$  characterize portfolios of  $R_{t+1}$  and  $(R'_{t+1}, r'_{t+1})'$ , respectively, whereas  $\alpha_J(\eta)$  and  $\Sigma_{\varepsilon\varepsilon}$  follow from a regression of  $r_{t+1}$  on  $R_{t+1}$ , and measure the performance of  $r_{t+1}$  relative to  $R_{t+1}$ . Stated differently, whereas Sharpe ratios can be used to compare the performance of different portfolios, Jensen's alpha gives the potential improvement in performance when the additional assets are included in the portfolio. The hypotheses of intersection and spanning imply that Jensen's alpha,  $\alpha_J(\eta)$ , is zero for one or for all values of  $\eta$  respectively. Therefore, if there is intersection (spanning) then there is no improvement in the Sharpe measure possible by including the additional assets  $r_{t+1}$  in the investors portfolio.

## 5.2 Changes in optimal portfolio weights

The performance measures and the intersection regressions discussed above can also be used to infer the changes in optimal portfolio holdings when adding the assets  $r_{t+1}$ . In this section we will show that given the initial mean-variance efficient portfolio of the benchmark assets and the OLS-estimates of the regression parameters in (20), it is straightforward to determine the new optimal portfolio weights. Some of the results presented in this section are also presented in Stevens (1998). In order to derive the optimal portfolio weights from the regression results, consider the mean-variance efficient portfolio for the extended set  $(R_{t+1}, r_{t+1})$  for a given value of  $\eta$ :

$$w^* = \gamma^{-1} \Sigma^{-1} (\mu - \eta \iota).$$

Substituting the partitioned inverse as given in (52) in the expression for  $w^*$  gives that the optimal portfolio weights for the new assets,  $w_r^*$ , can be written as

$$\begin{aligned} w_r^* &= \gamma^{-1} \Sigma_{\varepsilon\varepsilon}^{-1} ((\mu_r - \beta \mu_R) - (\iota_N - \beta \iota_K) \eta) \\ &= \gamma^{-1} \Sigma_{\varepsilon\varepsilon}^{-1} \alpha_J(\eta). \end{aligned} \tag{57}$$

Thus, the optimal portfolio weights  $w_r^*$  are determined by the vector of Jensen's alphas and the covariance matrix of the residuals of the OLS-regression of  $r_{t+1}$  on  $R_{t+1}$ .<sup>4</sup>

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<sup>4</sup>As an aside, in terms of volatility bounds, notice that  $w_r^* \gamma = -\alpha_r(1/\eta)$ , i.e., the elements of  $\alpha(v)$  in (5) that correspond to  $r_{t+1}$ . Thus if we want to know the minimum variance stochastic discount factor from  $(R_{t+1}, r_{t+1})$ , rather than from  $R_{t+1}$ , the projection coefficients corresponding to the additional assets  $r_{t+1}$  are given by  $-\Sigma_{\varepsilon\varepsilon}^{-1} \alpha_J(\eta)$ .

In deriving the new optimal portfolio weights, a problem in (57) is that the coefficient of risk aversion  $\gamma$  is present. Notice that this is a different coefficient than the one that appears in the optimal portfolio  $\tilde{w}_R^*$  of the smaller set  $R_{t+1}$ :

$$\tilde{w}_R^* = \tilde{\gamma}_R^{-1} \Sigma_{RR}^{-1} (\mu_R - \eta \iota_K),$$

where we now also add a  $\sim$  to indicate that a variable refers to the set of benchmark assets  $R_{t+1}$  only. It is only the zero-beta return  $\eta$  that is the same in both problems, since we test whether there is intersection for a fixed value of  $\eta$ . Similarly, the expected returns on the portfolios  $\tilde{w}_R^*$  and  $w^*$  are different, and we indicate these with  $\tilde{m}_R$  and  $m$  respectively, i.e.,  $\tilde{m}_R \equiv \tilde{w}_R^{*\prime} \mu_R$ , and  $m \equiv w^{*\prime} \mu$ . In order to substitute out the risk aversion parameter  $\gamma$ , note that

$$\begin{aligned} \gamma &= B - \eta A = B_R - \eta A_R + \alpha_J(\eta)' \Sigma_{ee}^{-1} (\iota_N - \beta \iota_K) \\ &= \tilde{\gamma}_R + \alpha_J(\eta)' \Sigma_{ee}^{-1} (\iota_N - \beta \iota_K), \end{aligned}$$

and that

$$\tilde{\gamma}_R = \frac{\tilde{m}_R - \eta}{\tilde{w}_R^{*\prime} \Sigma_{RR} \tilde{w}_R^*} = \frac{\theta_R(\eta)^2}{\tilde{m}_R - \eta}.$$

Using these latter two expressions, the optimal portfolio weights  $w_r^*$  can be expressed as

$$w_r^* = \left( \frac{\tilde{m}_R - \eta}{\theta_R(\eta)^2 + (\tilde{m}_R - \eta) \alpha_J(\eta)' \Sigma_{ee}^{-1} (\iota_N - \beta \iota_K)} \right) \Sigma_{ee}^{-1} \alpha_J(\eta). \quad (58)$$

The advantage of (58) is that it contains only variables that either result from the initial optimal portfolio  $\tilde{w}_R^*$ , or from a regression of  $r_{t+1}$  on  $R_{t+1}$ .

Along the same lines it is straightforward to show that the new optimal weights  $w_R^*$  are given by

$$w_R^* = \left( \frac{\theta_R(\eta)^2}{\theta_R(\eta)^2 + (\tilde{m}_R - \eta) \alpha_J(\eta)' \Sigma_{ee}^{-1} (\iota_N - \beta \iota_K)} \right) \tilde{w}_R^* - \beta' w_r^*. \quad (59)$$

Again, this expression only depends on characteristics of the old portfolio,  $\tilde{w}_R^*$ , and the regression output. Therefore, given the initial mean-variance efficient portfolio  $\tilde{w}_R^*$  of the benchmark assets and the OLS-estimates of the regression in (20), (58) and (59) answer the question how to adjust the portfolio in order to obtain the new mean-variance efficient portfolio  $w^*$ .



In order to give an interpretation of the new portfolio weights in (58) and (59), it is useful to rewrite them in the following way:<sup>5</sup>

$$w_r^* = \frac{m - \eta}{\theta(\eta)^2} \Sigma_{\epsilon\epsilon}^{-1} \alpha_J(\eta), \quad (60)$$

and

$$w_R^* = \frac{\theta_R(\eta)^2}{\theta(\eta)^2} \frac{m - \eta}{\tilde{m}_R - \eta} \tilde{w}_R - \beta' w_r^*. \quad (61)$$

If there is only one new asset, i.e.,  $N = 1$ , Equation (60) first of all shows that  $\alpha_J(\eta)$  determines the sign of the new portfolio weight  $w_r^*$  (given that  $m - \eta > 0$ ): if Jensen's alpha is positive (negative) the investor can improve the performance of his portfolio by taking long (short) positions in the new asset. When there is more than one new asset, the sign of the portfolio weights is not only determined by the sign of Jensen's alpha, but also by the inverse of the covariance matrix of  $\varepsilon_{t+1}$ . If the mean-variance frontier is not strongly affected by the introduction of the new assets, then  $(\theta_R(\eta)^2/\theta(\eta)^2)(m - \eta)/(\tilde{m}_R - \eta) \approx 1$ , and the coefficients  $\beta$  show which of the old assets are replaced by the new assets.

Finally, notice that we did not consider a risk free asset. The portfolio weights given above are for the tangency portfolio when the zero-beta rate is  $\eta$ . If a risk free asset is available, we can replace  $\eta$  with  $R^f$  in (60) and (61) and these equations still give the portfolio weights for the tangency portfolio. The new tangency portfolio has an expected return equal to  $m$ , whereas the old tangency portfolio has an expected return  $\tilde{m}_R$ . Notice though, that in case a risk free asset is available it is easy to shift funds between the tangency portfolio and the risk free asset and to let the expected portfolio return vary. For practical purposes, the interest may be in the new portfolio  $w^*$  that has the same expected return as the old portfolio. Given that there is a risk free asset available, this is easily achieved by letting  $m - R^f = \tilde{m}_R - R^f$ . In this case Equations (60) and (61) simplify to

$$w_r^* = \frac{m - R^f}{\theta^2} \Sigma_{\epsilon\epsilon}^{-1} \alpha_J \quad (62)$$

and

$$w_R^* = \frac{\theta_R^2}{\theta^2} \tilde{w}_R - \beta' w_r^*. \quad (63)$$

<sup>5</sup>Here we use the fact that  $\theta_R(\eta)^2/(\tilde{m}_R - \eta) = A_R - \eta B_R$ , and that  $A_R - \eta B_R + \alpha_J(\eta)' \Sigma_{\epsilon\epsilon}^{-1} (\iota_N - \beta \iota_K) = A - \eta B$ .

Notice that here it is not necessarily the case that the weights in  $w_r^*$  and  $w_R^*$  sum to one. The investor will have to borrow or lend a fraction  $(1 - \iota_K' w_R^* - \iota_N' w_r^*)$  to achieve an expected portfolio return equal to  $m$ .

### 5.3 Interpretation of spanning and intersection tests in terms of performance measures

Finally, we want to relate the Wald test-statistics presented in Section 3 to the performance measures discussed above. It will be shown that these test-statistics can be expressed as changes in maximum Sharpe ratios of  $R_{t+1}$  and  $(R_{t+1}, r_{t+1})$  respectively. Therefore, they have a clear economic interpretation. In order to interpret the test-statistics for intersection and spanning in terms of performance measures, recall the basic regression model in (20):

$$r_{t+1} = \alpha + \beta R_{t+1} + \varepsilon_{t+1},$$

where intersection for a given value of  $\eta$  means that

$$\alpha_J(\eta) = \alpha + (\beta \iota_K - \iota_N) \eta = 0.$$

Thus, the restrictions on the regression coefficients that are imposed by the hypothesis of intersection have a natural interpretation in terms of Jensen's alphas, and - as noted before - testing whether there is intersection for  $\eta$ , is equivalent to testing whether Jensen's alpha is zero. Testing for spanning is of course equivalent to testing whether the Jensen's alphas are zero for all values of  $\eta$ .

It can be shown that the test statistics for intersection and spanning,  $\xi_W^{int}$  and  $\xi_W^{span}$ , presented in Section 3.4, can also be interpreted in terms of Jensen's alphas and Sharpe ratios. To see this, start again from the specification of the regression equation in (23):

$$r_{t+1} = \left( I_N \otimes \begin{pmatrix} 1 & R_{t+1}' \end{pmatrix} \right) b + \varepsilon_{t+1}.$$

Note that (using partitioned inverses) the asymptotic covariance matrix of the OLS-estimates of  $b$ ,  $\hat{b}$  in (23) is given by

$$\begin{aligned} & \Sigma_{\varepsilon\varepsilon} \otimes \begin{pmatrix} 1 & \mu_R' \\ \mu_R & E[R_t R_t'] \end{pmatrix}^{-1} \\ = & \Sigma_{\varepsilon\varepsilon} \otimes \begin{pmatrix} 1 + \mu_R' \Sigma_{RR}^{-1} \mu_R & -\mu_R' \Sigma_{RR}^{-1} \\ -\Sigma_{RR}^{-1} \mu_R & \Sigma_{RR}^{-1} \end{pmatrix}. \end{aligned} \quad (64)$$

Straightforward algebra shows that premultiplying (64) with  $H(\eta)_{int}$  and postmultiplying with  $H(\eta)'_{int}$  as defined in (25), yields

$$Var[\hat{\alpha}_J(\eta)] = \Sigma_{\epsilon\epsilon}(1 + \theta_R(\eta)^2), \quad (65)$$

where the Sharpe ratio  $\theta_R(\eta)$  was defined in (55). Since from the analysis above we know that the term  $h(\eta)_{int}$  as defined in (25) equals  $\alpha_J(\eta)$ , (56) can be used to rewrite the test statistic for intersection,  $\xi_W^{int}$ , as

$$\xi_W^{int} = T \frac{\hat{\alpha}_J(\eta)' \hat{\Sigma}_{\epsilon\epsilon}^{-1} \hat{\alpha}_J(\eta)}{1 + \hat{\theta}_R(\eta)^2} = T \left( \frac{1 + \hat{\theta}(\eta)^2}{1 + \hat{\theta}_R(\eta)^2} - 1 \right), \quad (66)$$

where  $\hat{\theta}_R(\eta)$ ,  $\hat{\theta}(\eta)$ , and  $\hat{\alpha}_J(\eta)$  are the sample Sharpe ratios and Jensen's alpha respectively. Equation (66) is a well known result from, e.g., Jobson & Korkie (1982) and Gibbons, Ross & Shanken (1989). It clearly shows that the Wald test statistic for intersection can easily be interpreted as the percentage increase in squared Sharpe ratios scaled by the sample size. Under the null-hypothesis that there is intersection,  $\theta(\eta) = \theta_R(\eta)$  and the increase of the sample Sharpe ratios scaled by the sample size  $T$  (as in (66)) will asymptotically have a  $\chi^2_{(N)}$ -distribution.<sup>6</sup>

For the spanning test-statistic, a similar interpretation can be given. Let  $\eta_R^0$  denote the expected return on the global minimum variance portfolio of  $R_{t+1}$ , i.e.,  $\eta_R^0 = B_R/A_R$ , and let the variance of this portfolio be given by  $(\sigma_R^0)^2$ . Similarly, let  $(\sigma^0)^2$  be the global minimum variance of  $(R_{t+1}, r_{t+1})$ . It is shown in Appendix C that the Wald test-statistic for spanning,  $\xi_W^{span}$ , can be written as

$$\xi_W^{span} = T \left( \frac{1 + \hat{\theta}(\hat{\eta}_R^0)^2}{1 + \hat{\theta}_R(\hat{\eta}_R^0)^2} - 1 \right) + T \left( \frac{(\hat{\sigma}_R^0)^2}{(\hat{\sigma}^0)^2} - 1 \right). \quad (67)$$

This shows that the spanning test-statistic consists of two parts. The first part is similar to the test-statistic for intersection in (66) and is determined by a change in Sharpe ratios. The Sharpe ratios in (67) are for a zero-beta rate equal to the (in-sample) expected return on the global minimum

<sup>6</sup> Gibbons, Ross & Shanken (1989) study the small sample properties of this test statistic in case there is a risk free asset, as well as the distribution under the alternative hypothesis. Kandel & Stambaugh (1987) and Shanken (1987) extend their results to the case where the mean-variance efficient benchmark portfolio (or intersection portfolio) can not be observed but has a given correlation with observed proxy portfolio.

variance portfolio however, and therefore are the slopes of the asymptotes of the mean-variance frontier. Notice that the slope of the upper limb of the frontier is simply the negative of the slope of the lower limb of the frontier, and therefore, the squared Sharpe ratios for those two extremes are the same. The first term of the spanning test-statistic in a sense measures whether there is intersection at the most extreme points of the frontier (i.e., whether there is a limiting form of intersection if we go sufficiently far up or down the frontier). The second term of the statistic in (67) is determined by the change in the global minimum variance of the portfolios, and measures whether the point most to the left on the frontier changes or not. Put differently, the first term measures whether there is intersection for a mean-variance investor with a very small risk aversion ( $\gamma = 0$ ), while the second term measures whether there is intersection for a mean-variance investor with a very high risk aversion ( $\gamma \rightarrow \infty$ ). Note that in the second term the old global minimum variance appears in the numerator and the new global minimum variance in the denominator, since this variance can only decrease as assets are added to the portfolio. Therefore, both terms in (67) are always larger than or equal to one. Jobson & Korkie (1989) derive a similar result for a likelihood ratio test for spanning.

## 6 Specification error bounds and intersection

As in the previous section, in this section the focus will be on deviations from intersection rather than on intersection itself. In a recent paper Hansen & Jagannathan (1997) analyze *specification errors* in stochastic discount factor models which, in some special cases, can be interpreted as deviations from intersection. They derive bounds on the magnitude of these specification errors which we will apply to models for futures risk premia in paper 6. Therefore, the analysis in this section also serves as an introduction to paper 6.

Recall from the discussion in Section 2.1 that each asset pricing models assigns a particular function to the pricing kernel  $M_{t+1}$ . Hansen & Jagannathan (1997) note that the pricing kernels implied by most asset pricing models do not yield correct asset prices, either because the asset pricing model can only be viewed as an approximation, or because of measurement error. Measurement errors are for instance often considered to be an important problem in measuring consumption and testing consumption based



asset pricing models. Therefore, the pricing kernel implied by an asset pricing model will in general only serve as a *proxy* stochastic discount factor, that will not yield the correct prices or expected payoffs of the assets under consideration. The interest of Hansen & Jagannathan is in the least squares distance between such a proxy stochastic discount factor and the set of valid stochastic discount factors. They derive a lower bound on this distance, the *specification error bound*, as a measure of how well the model performs. These specification error bounds will be derived formally below and it will also be shown that these bounds have a clear economic interpretation in terms of maximum pricing errors or maximum expected payoff errors implied by the asset pricing model. Hansen, Heaton, & Luttmer (1995) derive the limiting distribution for the corresponding estimator of the specification error bounds.

It turns out that if we take the minimum variance stochastic discount factor for the subset  $R_{t+1}$  as a proxy stochastic discount factor for the larger set of assets  $(R_{t+1}, r_{t+1})$ , we can interpret the specification error bounds in terms of mean-variance intersection and the performance measures discussed in the previous section. In particular, provided that both the proxy stochastic discount factor and the discount factors that price  $R_{t+1}$  and  $r_{t+1}$  correctly have the same expectation  $v$ , the squared specification error bound scaled by  $v$  turns out to be equal to the difference between the maximum squared Sharpe ratio implied by the set  $R_{t+1}$  and the maximum squared Sharpe ratio implied by  $(R_{t+1}, r_{t+1})$ . This also allows us to interpret the specification errors in terms of mean-variance portfolio choice again. Given that a mean-variance investor is aware of the fact that a portfolio chosen from the subset  $R_{t+1}$  is suboptimal relative to a portfolio chosen from the larger set  $(R_{t+1}, r_{t+1})$ , the specification error bound gives an estimate of the extent to which the portfolio is suboptimal in terms of Sharpe ratios.

## 6.1 Specification error bounds

As noted above, in Hansen & Jagannathan (1997) the interest is in *proxy* stochastic discount factors, denoted by  $y_{t+1}$ , that assign approximate prices to portfolio payoffs. For instance, the CAPM implies that the proxy is of the form  $a + bR_{t+1}^m$ , with  $R_{t+1}^m$  the return on the market portfolio. As before, let  $R_{t+1}^p$  be the return on some portfolio, not necessarily mean-variance efficient, such that  $w^p \iota_K = 1$ . The expected price assigned to such a portfolio by a

proxy stochastic discount factor will be denoted by  $\pi^a(R_{t+1}^p)$ :

$$E[y_{t+1}R_{t+1}^p] = \pi^a(R_{t+1}^p). \quad (68)$$

Of course, valid stochastic discount factors  $M_{t+1}$  would assign a price  $\pi(R_{t+1}^p) = 1$  to any portfolio  $w^p$  that satisfies  $w^p \iota_K = 1$ . Because the proxy  $y_{t+1}$  may be derived from an asset pricing model that is strictly speaking not valid, or because the proxy may be measured with error, the prices assigned by the proxy,  $\pi^a(R_{t+1}^p)$ , will in general not be equal to one. The discussion here is somewhat restrictive because we only consider payoffs that are returns, i.e., payoffs with (correct) prices equal to one. Hansen & Jagannathan (1997) take more general payoffs  $x_{t+1}$  with current prices  $q_t$ . Given that we want to establish the relation between specification errors and mean-variance intersection, the use of returns is not very restrictive however. Moreover, the results derived below can easily be adjusted to the results of Hansen & Jagannathan along the lines of Section 4.1, where we incorporated conditioning information by allowing for payoffs  $z_t \otimes R_{t+1}$  with current prices  $q_t$ .

A second way in which the results here are somewhat more restrictive than the ones in Hansen & Jagannathan (1997) is that we will always consider valid stochastic discount factors  $M(v)_{t+1}$  that have the same expectation as the proxy  $y_{t+1}$ , i.e.,  $v = E[y_{t+1}]$ . This may be considered as restrictive, since this assumption in fact requires that the proxy assigns the correct price to the risk free payoff, if it exists. Once more, given that the interest here is in the relation with mean-variance intersection in the absence of a risk free asset, and given that we always defined intersection for a known value of  $v$ , this is not restrictive for our purposes.

The problem addressed in Hansen & Jagannathan (1997) is to derive a lower bound  $\delta$  on the distance between  $y_{t+1}$  and the set of stochastic discount factors that price  $R_{t+1}$  correctly, which we denote as  $\mathcal{M}$ :

$$\delta = \min_{\{M_R(v)_{t+1} \in \mathcal{M}\}} \|y_{t+1} - M_R(v)_{t+1}\|, \quad (69)$$

where  $\|x_{t+1}\| \equiv E[x_{t+1}^2]^{1/2}$ . Because  $y_{t+1}$  and  $M_R(v)_{t+1}$  have the same expectation, the distance between  $y_{t+1}$  and  $M_R(v)_{t+1}$  in (69) is equal to the standard deviation of  $y_{t+1} - M_R(v)_{t+1}$ , i.e.,  $\|y_{t+1} - M_R(v)_{t+1}\| = \sigma(y_{t+1} - M_R(v)_{t+1})$ . We will denote the stochastic discount factor that solves (69) by  $\tilde{m}_R(v)_{t+1}$ . Thus,  $\tilde{m}_R(v)_{t+1}$  is the stochastic discount factor that prices  $R_{t+1}$  correctly and that is closest to  $y_{t+1}$  in a least squares sense.

To solve the problem in (69), consider the least squares projections of  $y_{t+1}$  and  $M_R(v)_{t+1}$  on  $R_{t+1}$  and a constant:

$$\begin{aligned}\hat{y}_{t+1} &= \text{Proj}(y_{t+1} \mid 1, R_{t+1}) = v + \zeta(v)'(R_{t+1} - \mu_R), \\ y_{t+1} &= \hat{y}_{t+1} + u_{t+1},\end{aligned}\tag{70}$$

and

$$\begin{aligned}m_R(v)_{t+1} &= \text{Proj}(M_R(v)_{t+1} \mid 1, R_{t+1}) = v + \alpha(v)'(R_{t+1} - \mu_R), \\ M_R(v)_{t+1} &= m_R(v)_{t+1} + w_{t+1},\end{aligned}\tag{71}$$

where  $m_R(v)_{t+1}$  is the minimum variance stochastic discount factor derived in Section 2.1, and  $\alpha(v)$  is defined in (5). The projection coefficients in (70) are given by  $\Sigma_{RR}^{-1}\Sigma_{Ry}$ , with  $\Sigma_{Ry}$  the  $K \times 1$ -vector of covariances between  $R_{t+1}$  and  $y_{t+1}$ . Noting that  $\|y_{t+1} - M_R(v)_{t+1}\|^2 = \text{Var}[y_{t+1} - M_R(v)_{t+1}]$ , it easily follows that

$$\begin{aligned}\text{Var}[y_{t+1} - M_R(v)_{t+1}] &= \text{Var}[\hat{y}_{t+1} - m_R(v)_{t+1}] + \text{Var}[u_{t+1} - w_{t+1}] \\ &\geq \text{Var}[\hat{y}_{t+1} - m_R(v)_{t+1}].\end{aligned}$$

Because  $\hat{y}_{t+1} - m_R(v)_{t+1} = y_{t+1} - (m_R(v)_{t+1} + u_{t+1})$  and  $u_{t+1}$  is orthogonal to  $R_{t+1}$ , this lower bound on the variance of  $y_{t+1} - M_R(v)_{t+1}$  is attainable for the stochastic discount factor

$$\tilde{m}_R(v)_{t+1} = m_R(v)_{t+1} + u_{t+1},\tag{72}$$

and we have that

$$\delta^2 = \text{Var}[y_{t+1} - \tilde{m}_R(v)_{t+1}].\tag{73}$$

A more detailed characterization of  $\tilde{m}_R(v)_{t+1}$  and  $\delta$  will be given in the following section. For this moment, note that subtracting the variable  $y_{t+1} - \tilde{m}_R(v)_{t+1}$  from the proxy  $y_{t+1}$  yields a valid stochastic discount factor. Therefore, as noted by Hansen & Jagannathan (1997),  $y_{t+1} - \tilde{m}_R(v)_{t+1}$  is the smallest adjustment in a least squares sense that is necessary to make  $y_{t+1}$  a valid stochastic discount factor, and  $\delta$  is a measure of the magnitude of this adjustment.

Hansen & Jagannathan also show that  $\delta$  can be interpreted as a maximum pricing error. In order to do so, let  $\omega$  denote a position in  $R_{t+1}$  that does not necessarily satisfy the requirement  $\omega' \iota_K = 1$ , i.e.,  $\omega$  is in general not a

portfolio. Denote the payoff of such a position as  $R(\omega)_{t+1} = \omega' R_{t+1}$  and note that the correct price of such a position is

$$E[\omega' R_{t+1} M_R(v)] = \pi(R(\omega)_{t+1}) = \omega' \iota_K,$$

whereas the price assigned by the proxy  $y_{t+1}$  is  $\pi^a(R(\omega)_{t+1})$ . The pricing error of the proxy  $y_{t+1}$  is therefore  $\pi^a(R(\omega)_{t+1}) - \pi(R(\omega)_{t+1})$ , and Hansen & Jagannathan show that  $\delta$  provides an upper bound on the absolute value of this pricing error, for positions that have a unit norm:

$$\delta = \max_{R(\omega)_{t+1}, \|R(\omega)_{t+1}\|=1} | \pi^a(R(\omega)_{t+1}) - \pi(R(\omega)_{t+1}) |.$$

Thus, by looking at a particular class of positions, i.e., positions with a unit norm,  $\delta$  can be interpreted as the maximum pricing error assigned by the proxy to the payoffs of those unit norm positions.

A more intuitive interpretation can be given if we consider errors in expected payoffs, or expected returns, rather than pricing errors. Recall that a valid stochastic discount factor assigns the correct expected return to a one-dollar investment in portfolio  $w^p$  (for which, by definition,  $w^{p'} \iota = 1$ ) which, using equation (3), can be written as

$$E[R_{t+1}^p] = \frac{1}{v} - \frac{\text{Cov}[M_R(v)_{t+1}, R_{t+1}^p]}{v},$$

i.e., as one over the expectation of the pricing kernel, which equals the risk free rate if it exists, plus a risk term which is determined by the covariance of the portfolio return and the pricing kernel. Observe that use of the proxy, that also has expectation  $v$ , would give an approximate expected return  $E^a[R_{t+1}^p]$  for a one-dollar investment in  $w^p$  that in general differs from  $E[R_{t+1}^p]$ , because the covariance of the proxy with the portfolio return will be different from the covariance of a valid stochastic discount factor with the portfolio return, i.e.:

$$E^a[R_{t+1}^p] = \frac{1}{v} - \frac{\text{Cov}[y_{t+1}, R_{t+1}^p]}{v}.$$

From these relations we define the *expected return error*

$$E^a[R_{t+1}^p] - E[R_{t+1}^p] = \frac{\text{Cov}[M_R(v)_{t+1} - y_{t+1}, R_{t+1}^p]}{v}, \quad (74)$$



for which the Cauchy-Schwarz inequality implies that

$$|E^a[R_{t+1}^p] - E[R_{t+1}^p]| \leq \frac{\sigma(y_{t+1} - M_R(v)_{t+1})\sigma(R_{t+1}^p)}{v}.$$

Since this inequality holds for all valid stochastic discount factors  $M_R(v)_{t+1}$ , it also holds for the stochastic discount factor that solves (69),  $\tilde{m}_R(v)_{t+1}$ , implying

$$|E^a[R_{t+1}^p] - E[R_{t+1}^p]| \leq \frac{\delta\sigma(R_{t+1}^p)}{v}.$$

Since for a given value of  $v$ , the Sharpe ratio is defined as  $Sh(R_{t+1}^p) \equiv (E[R_{t+1}^p] - 1/v)/\sigma(R_{t+1}^p)$ , and the approximate Sharpe ratio, i.e., the Sharpe ratio according to the proxy  $y_{t+1}$ , as  $Sh^a(R_{t+1}^p) \equiv (E^a[R_{t+1}^p] - 1/v)/\sigma(R_{t+1}^p)$ , this can be rewritten as

$$|Sh^a(R_{t+1}^p) - Sh(R_{t+1}^p)| \leq \frac{\delta}{v}. \quad (75)$$

Thus, using errors in expected returns rather than errors in assigned prices, the specification error bound  $\delta$  scaled by the expectation of the proxy has a very clear interpretation in terms of Sharpe ratios. For any portfolio  $w^p$  formed from the assets in  $R_{t+1}$ , the absolute difference between the approximate Sharpe ratio assigned to the portfolio returns by  $y_{t+1}$  and the actual Sharpe ratio of the portfolio can never exceed the scaled specification error bound  $\delta/v$ . This interpretation is also somewhat easier than the one given for the expected payoff error in Hansen & Jagannathan (1997), where they focus on the maximum error in expected payoffs for positions  $\omega$  with unit standard deviation.

## 6.2 The relation between specification error bounds and intersection

The purpose of this section is to show that there is a close relation between intersection and a special case of the specification error bounds. In particular, if the interest is in stochastic discount factors that price the returns  $(R_{t+1}, r_{t+1})$  correctly and we choose for the proxy  $y_{t+1}$  the minimum variance stochastic discount factor based on the subset  $R_{t+1}$ ,  $m_R(v)_{t+1}$ , the specification error bound can simply be expressed as a deviation from intersection, as was the case with the performance measures discussed in Section 5. To

show this, let us first give a more precise characterization of  $\tilde{m}(v)_{t+1}$  and  $\delta$  than given in (72) and (73).

Recall that  $\tilde{m}_R(v)_{t+1}$  is given by  $m_R(v)_{t+1} + u_{t+1}$ , where  $u_{t+1} = y_{t+1} - \hat{y}_{t+1}$ . Using (71) and (70), this implies for  $\tilde{m}_R(v)_{t+1}$ :

$$\begin{aligned}\tilde{m}_R(v)_{t+1} &= v + \alpha(v)'(R_{t+1} - \mu_R) + y_{t+1} - \{v + \zeta(v)'(R_{t+1} - \mu_R)\} \\ &= y_{t+1} + (\alpha(v) - \zeta(v))'(R_{t+1} - \mu_R) \\ &= y_{t+1} + \{(\iota_K - v\mu_R) - \Sigma_{Ry}\}'\Sigma_{RR}^{-1}(R_{t+1} - \mu_R),\end{aligned}\quad (76)$$

and for  $\delta^2$ :

$$\delta^2 = \{(\iota_K - v\mu_R) - \Sigma_{Ry}\}'\Sigma_{RR}^{-1}\{(\iota_K - v\mu_R) - \Sigma_{Ry}\}. \quad (77)$$

For further reference it is useful to define the vector  $\kappa(v)$  as

$$\kappa(v) = \alpha(v) - \zeta(v) = \Sigma_{RR}^{-1}\{(\iota_K - v\mu_R) - \Sigma_{Ry}\}. \quad (78)$$

Notice that the expressions for  $\kappa(v)$  and  $\delta^2$  given here differ slightly from the ones given in Hansen & Jagannathan (1997) because we explicitly included a constant in the projections of  $M(v)_{t+1}$  and  $y_{t+1}$  on  $R_{t+1}$ .

The expressions for  $\tilde{m}_R(v)_{t+1}$  and  $\delta^2$  in (76) and (77) provide a basis to relate the specification error bounds to intersection. In case of intersection the interest is in stochastic discount factors that price both  $R_{t+1}$  and  $r_{t+1}$ , i.e., in  $M(v)_{t+1}$ . Therefore, in the expressions (76) and (77) we should leave out all the  $R$ -subscripts, replace  $R_{t+1}$  with the vector  $(R'_{t+1} \ r'_{t+1})'$ , and note that all vectors and matrices have dimension  $K + N$  rather than  $K$ . As before, with intersection we want to know if the minimum variance stochastic discount factor based on  $R_{t+1}$  only,  $m_R(v)_{t+1}$  can be used to price both  $R_{t+1}$  and  $r_{t+1}$ . In terms of specification errors this means that we want to use  $m_R(v)_{t+1}$  as a proxy  $y_{t+1}$  for the stochastic discount factors  $M(v)_{t+1}$ . Also, in the spirit of the previous section, when using  $m_R(v)_{t+1}$  as a proxy, we recognize beforehand that  $m_R(v)_{t+1}$  will not assign the correct prices to  $r_{t+1}$ , but the interest is in the extent to which the assigned prices are wrong, i.e., the extent to which there are deviations from intersection, as measured by  $\delta$ .

Recall that the proxy  $y_{t+1} = m_R(v)_{t+1}$  is now given by

$$\begin{aligned}y_{t+1} &= m_R(v)_{t+1} = v + \alpha_R(v)'(R_{t+1} - \mu_R), \\ \alpha_R(v) &= \Sigma_{RR}^{-1}(\iota_K - v\mu_R).\end{aligned}$$

Substituting these expressions into (76) and (77), properly adjusted for the fact that the interest is now in stochastic discount factors that price both  $R_{t+1}$  and  $r_{t+1}$ , straightforward algebra shows that

$$\begin{aligned}\delta^2 &= \{(\iota_N - v\mu_r) - \Sigma_{rR}\Sigma_{RR}^{-1}(\iota_K - v\mu_R)\}'\Sigma_{\varepsilon\varepsilon}^{-1}\{(\iota_N - v\mu_r) - \Sigma_{rR}\Sigma_{RR}^{-1}(\iota_K - v\mu_R)\} \\ &= v^2\alpha_J(1/v)'\Sigma_{\varepsilon\varepsilon}^{-1}\alpha_J(1/v),\end{aligned}\tag{79}$$

or

$$\frac{\delta}{v} = \{\theta(1/v)^2 - \theta_R(1/v)^2\}^{1/2},$$

where  $\Sigma_{\varepsilon\varepsilon}$  is the covariance matrix of the residuals  $\varepsilon_{t+1}$  from a regression of  $r_{t+1}$  on  $R_{t+1}$  and a constant. Also, the stochastic discount factor closest to  $y_{t+1}$  is now given by

$$\widetilde{m}(v)_{t+1} = m_R(v)_{t+1} + v\alpha_J(1/v)'\Sigma_{\varepsilon\varepsilon}^{-1}\varepsilon_{t+1} = m(v)_{t+1}.\tag{80}$$

Thus, if we want to use the stochastic discount factor that is on the volatility bound of  $R_{t+1}$ , as a proxy stochastic discount factor for the larger set  $(R_{t+1}, r_{t+1})$ , then the valid discount factor that is closest to  $m_R(v)_{t+1}$  is the discount factor with the same expectation  $v$  that is on the volatility bound of  $(R_{t+1}, r_{t+1})$ . Therefore,  $\delta$  is the least squares distance between two stochastic discount factors that are on the volatility bounds of  $(R_{t+1}, r_{t+1})$  and its subset  $R_{t+1}$  respectively, and is a straightforward measure of the deviation from intersection, which shows the close relation between this special case of the specification error bound and intersection. This relationship also follows from (79), which shows that  $\delta$  is directly related the change in the maximum squared Sharpe ratios that can be attained with  $R_{t+1}$  and  $(R_{t+1}, r_{t+1})$  respectively. It also follows that  $\delta$  measures the difference between the variances of the two minimum variance kernels:  $\delta = \text{Var}[m(v)_{t+1}] - \text{Var}[m_R(v)_{t+1}]$ .

An estimate of  $\delta^2$  can easily be obtained from the sample equivalent of (77), which we will denote by  $\widehat{\delta}^2$ . If the interest is in whether or not there is intersection, then we want to know whether or not  $\delta = 0$ , and this hypothesis can easily be tested as outlined in Section 3. From the expression in (79) and the discussion in previous sections, it follows that under the null hypothesis that  $\delta = 0$ ,

$$T \frac{\widehat{\delta}^2}{v^2(1 + \widehat{\theta}_R(1/v)^2)} \sim \chi_N^2.\tag{81}$$

In case of specification errors however, the interest is in the case where  $\delta$  is strictly positive rather than zero. For that case the limiting distribution of  $\hat{\delta}$  is derived in Hansen, Heaton, & Luttmer (1995).

Once we concede that  $y_{t+1} = m_R(v)_{t+1}$  is not a valid stochastic discount factor for  $(R_{t+1}, r_{t+1})$ , we want to have a measure of the difference between  $m_R(v)_{t+1}$  and the valid stochastic discount factor that is closest to it,  $m(v)_{t+1}$ . The specification error bound  $\delta$  is one such measure, allowing us to make statements about how good or how bad the proxy performs. The fact that  $\delta^2$  is equal to the change in maximum Sharpe ratios, makes the measure  $\delta$  also useful in terms of the optimal portfolio choice for a mean-variance investor. Recall that a mean-variance investor that initially only invests in  $R_{t+1}$  can improve his Sharpe ratio from  $\theta_R(1/v)$  to  $\theta(1/v)$  by including  $r_{t+1}$  in his portfolio. Given that there is no intersection between the mean-variance frontiers of  $R_{t+1}$  and  $(R_{t+1}, r_{t+1})$ ,  $\hat{\delta}$  provides an estimate for the potential increase in Sharpe ratios. Notice though that such an estimate can also be derived directly from the Wald test-statistic for intersection.

## 7 Applications

In this section we will discuss some applications of the theoretical framework outlined in the previous sections to some problems that have recently received a lot of attention in the finance literature. These problems concern the diversification benefits of international investments and the efficiency of currency hedging, the diversification benefits of emerging markets, and the three-factor model that has recently been proposed by Fama & French (1996) to explain some well-known CAPM-anomalies. Because these problems are merely meant as an illustration we will not give a complete treatment of them, but only show how they relate to the concepts discussed in this paper. The applications that we discuss also show that understanding the relations between test-statistics for intersection and spanning, performance measures, efficient portfolio weights, and the coefficients in the spanning regression (20), can yield useful reinterpretations of many results that have been reported in the literature.



## 7.1 International diversification

It has often been argued that because correlations between stock returns are much lower between countries than within countries, there may be diversification benefits from investing in international stocks as opposed to domestic stocks only (see, e.g., Solnik (1991)). DeSantis (1995) analyses benefits from international diversification using a comprehensive dataset, that consists of monthly observations of the MSCI Indices over the period July 1973 until December 1992. DeSantis investigates whether it is useful for a US-investor to invest in Europe (Austria, Belgium, Denmark, France, Germany, Italy, the Netherlands, Norway, Spain, Sweden, Switzerland and the United Kingdom), in Pacific Basin countries (Australia, Hong Kong, Japan and Singapore) and in Canada. The risk free rate is taken to be equal to 0.62% per month, which is derived from the expectation of the kernel that is on the minimum of the volatility bound for the US only. The empirical results of DeSantis (1995) are for the null-hypothesis that the mean-variance frontier (volatility bound) of the US intersects the mean-variance frontier (volatility bound) of the US plus the set of European countries, the Pacific Basin countries, or a Global set (Europe + Pacific Basin + Canada). DeSantis reports the increase in the volatility bound, i.e., in  $\sigma(m(v)_{t+1})$ . However, from Equations (7) and (55) it can easily be seen that

$$\text{Var}[m(v_{t+1})] = v^2\theta(1/v)^2.$$

Thus, because of the duality between volatility bounds and mean-variance frontiers it is straightforward to obtain the increase in the Sharpe ratios from the results in DeSantis (1995). The tests used by DeSantis to analyze whether the shifts in the volatility bounds (mean-variance frontiers) are statistically significant are based on the GMM-test for overidentifying restrictions described at the end of Section 3.4. Because the reported Sharpe ratio for the US only is 0.089, it is also possible to derive the Wald test-statistic for intersection as in (66) from these results. The results reported by DeSantis, as well as the derived results in terms of Sharpe ratios and the Wald test for intersection, are reported in Table 1. These results are based on unhedged US-dollar based returns.

First of all, notice that because  $v$  is close to one, the reported change in the volatility bound ( $\Delta\text{bound}$ ) is close to the increase in the Sharpe ratio ( $\Delta\text{Sharpe}$ ). From an economic point of view, the reported changes in the Sharpe ratios are quite large, suggesting that there are large diversification

Table 1: Intersection tests for international diversification

The table presents intersection tests for international stock portfolios as reported by DeSantis (1995) as well as the implied changes in Sharpe ratios. The MSCI Index for the US is the benchmark. Europe = Austria, Belgium, Denmark, France, Germany, Italy, the Netherlands, Norway, Spain, Sweden, Switzerland and the United Kingdom; Pacific Basin = Australia, Hong Kong, Japan and Singapore; Global = Europe + Pacific Basin + Canada. Results are based on monthly unhedged US-dollar based returns for the period from July 1973 until December 1992. The intersection tests are based on  $v = 1.0062$ .  $\Delta\text{bound}$  is the change in the volatility of the minimum variance kernel with expectation  $v$ ;  $\Delta\text{Sharpe}$  is the change in the (maximum) Sharpe ratio for  $\eta = 1/v$ .

	$\Delta\text{bound}$	$\Delta\text{Sharpe}$	$p\text{-value (GMM)}$	Wald	$p\text{-value (Wald)}$
Europe	0.103	0.103	(0.847)	6.74	(0.875)
Pacific Basin	0.048	0.048	(0.744)	2.51	(0.643)
Global	0.135	0.135	(0.934)	9.85	(0.910)

benefits possible from international investments. However, the  $p$ -values associated with both the GMM-based tests and the Wald tests show that the increase in the Sharpe ratios are not statistically significant. Notice that in case of Europe for instance, the increase in the Sharpe ratio by 0.103 is obtained after adding 12 countries. Therefore, both the Wald test-statistic and the GMM test-statistic are asymptotically  $\chi^2_{12}$ -distributed under the null-hypothesis of intersection. With a sample size of 234 observations, the observed increase in Sharpe ratios is not sufficient to reject the hypothesis of spanning with this number of degrees of freedom.

Similar results can also be derived from Glen & Jorion (1993), who use monthly unhedged US-dollar based returns on the MSCI Indices in excess of the US T-Bill rate for the period January 1974 until December 1990. Although the main interest in Glen & Jorion (1993) is on the benefits of hedging the currency exposure of international stock and bond portfolios, from their results we can also derive some conclusions about the benefits of international diversification. For instance, from their summary statistics it follows that the (monthly) Sharpe ratio of US stocks is 0.079 for their sample period. When adding the MSCI Indices for the other G5 countries, Germany, Japan, the United Kingdom and France, the maximum attainable Sharpe ratio increases to 0.166. Since there are 204 observations, using (66),

this implies for the Wald test-statistic for intersection

$$W^{int} = 204 \left( \frac{1 + 0.166^2}{1 + 0.079^2} - 1 \right) = 4.33.$$

The  $p$ -value associated with this test is 0.363, implying that the increase in Sharpe ratios from 0.079 to 0.166 is again not statistically significant. This is also the case when both stocks and bonds from all countries are considered together. A mean-variance efficient portfolio of stocks and bonds from these five countries yields a Sharpe ratio of 0.184. If the null-hypothesis is that all these 10 securities are spanned by the MSCI Index for US stocks (plus T-Bills) only, then the Wald test-statistic is equal 5.61. Given that there are 9 securities added to the portfolio, the Wald test-statistic is asymptotically  $\chi^2_9$ -distributed. The  $p$ -value for the test-statistic of 5.61 is therefore 0.778, showing that even when both domestic and international stock and bond portfolios are added to the US stock-index, there is no (statistically) significant increase in the Sharpe ratio.

All these results are based on unhedged US-dollar based returns. As noted, in Glen & Jorion (1993) the interest is in the benefits of hedging currency risk associated with foreign investments. They show that there are significant diversification benefits, both statistically and economically, from including forward contracts in a portfolio of international bonds, or stocks and bonds, but not in a portfolio of international stocks only. For instance, for a US-investor that initially invests in the stocks and bonds of the five countries mentioned earlier, including forward contracts on the four currencies (German Mark, Japanese Yen, British Pound, and French Franc) causes an increase in the Sharpe ratio from 0.184 to 0.299. Thus, the null-hypothesis is that the mean-variance frontier of the 10 stock and bond indices intersects (spans) the mean-variance frontier of these same indices plus four currency forward contracts, when the risk free rate is equal to the one month US T-Bill rate. From the reported Sharpe ratios, the Wald test-statistic for this null-hypothesis 10.96 with a  $p$ -value of 0.027. In DeRoos, Nijman & Werker (1998b) it is shown how forward contracts can be included directly in the regression framework for testing for mean-variance spanning and intersection and how we can test whether or not hedging is beneficial for fixed portfolios rather than the portfolios considered by Glen & Jorion, where the optimal bond, stock and forward positions are chosen simultaneously.

Notice though, that Glen & Jorion (1993) assume that the investor will in any case choose his portfolio from the stocks and bonds of all these five coun-



tries. We already saw above that we can not reject the null hypothesis that the mean-variance frontier of the MSCI Index for US stocks intersects (spans) the mean-variance frontier of the stocks and bonds of all five countries. This suggests that the diversification benefits of the forward contracts may be much larger than the diversification benefits of the international stocks and bonds. It is therefore natural to ask if the US-investor can benefit from adding international stocks and bonds as well as the currency forwards to his portfolio of US stocks only. In other words, can we reject the null-hypothesis that the frontier of the MSCI Index for US stocks intersects (spans) the frontier of the stocks and bonds of all five countries *plus* the four forward contracts? This would induce an increase in the Sharpe ratio from 0.079 to 0.299, which is significant in economic terms. However, notice that this increase is obtained by adding 13 securities to the MSCI Index for the US. The Wald test-statistic for intersection of 16.87 is therefore asymptotically  $\chi^2_{13}$ -distributed, implying a *p*-value of 0.205. Therefore, this latter hypothesis can not be rejected.

Summarizing, although it is often claimed that international diversification leads to more efficient investment portfolios, based on the evidence reported in DeSantis (1995) and Glen & Jorion (1993) we can not reject the hypothesis that the mean-variance frontier of the MSCI Index of the US intersects the mean-variance frontier of this same index plus a number international stock and bond portfolios and forward contracts.

In economic terms, the increase in the Sharpe ratio that may be obtained from international diversification as reported by DeSantis (1995) and Glen & Jorion (1993) is often impressive. Given the number of observations in these studies and the number of securities that is added to the portfolio, these increases are not statistically significant however. On the other hand, for an investor who has invested in a portfolio of unhedged international stocks and bonds, adding forward contracts to hedge his currency exposure yields an increase in the Sharpe ratio that is both economically and statistically significant. Also, although not reported here, DeSantis (1995) shows that the hypothesis of intersection can be rejected when including managed portfolios, i.e., when incorporating conditional information (the lagged return on the World portfolio, the lagged dividend yield on the World portfolio and the term premium in the US bond market). Similarly, Glen & Jorion (1993) show that the benefits from currency hedging are much more profound when the hedging strategy is conditional on the forward premium (i.e., the interest differential between two countries).



## 7.2 Emerging markets

The results in the previous section showed that the benefits of international diversification to a US-investor, although often impressive in economic terms, are usually not statistically significant. However, in the studies of DeSantis (1995) and Glen & Jorion (1993) the focus is on portfolios consisting of investments in the US as well as a number of other well-developed equity markets, such as the German, Japanese and UK markets. The past twenty years have witnessed the emergence of many new equity markets in Europe, Latin America, Asia, the Mideast and Africa that offer new investment opportunities to investors. These emerging markets have been characterized by both high average returns and high volatility, but low correlations with equity returns in the developed markets. Therefore, although these emerging markets in themselves appear to provide risky investments, they may also provide substantial diversification benefits to US-investors.

For instance, Harvey (1995) reports an annualized average return of 20.36% for the Composite Index for emerging markets of the International Finance Corporation (IFC) over the period February 1985 until June 1992. The annualized standard deviation of this index is 24.70%. For the period February 1976 until June 1992 the average return on the MSCI World Index was 13.91%, and the standard deviation 14.36%. Moreover, the annualized average return for the individual emerging markets over the period February 1976 until June 1992 is in the range between 9.43% (Greece) and 71.79% (Argentina). The standard deviations during this period are in the range between 25.67% (Thailand) and 105.06% (Argentina). These statistics are all based on US dollar-based returns and show that the emerging markets are indeed characterized by high average returns and high volatility. As with respect to correlations, Harvey (1995) reports that the average cross country correlation in 18 developed markets is 0.41, whereas the average cross country correlation in 20 emerging markets is only 0.12. Furthermore, the average correlation between emerging markets and developed markets is 0.14, suggesting that there may be large diversification benefits from investing in emerging markets. This is confirmed by a comparison of the mean-variance frontiers of 18 developed countries and of 18 developed countries and 18 emerging markets, as presented in Harvey (1995). For the developed countries only, the global minimum-variance portfolio has a standard deviation of 13%. Adding the 18 emerging markets results in a global minimum-variance portfolio with a standard deviation of only 7%

Recall from Equation (67) in Section 5.3 that the Wald test-statistic for spanning can be decomposed into a part that is determined by the change in the global minimum variance and a part that is determined by the slope of the asymptotes along the frontier. These terms are always nonnegative and additive, so a lower bound for the Wald test-statistic for spanning can be derived by calculating the part in (67) that depends on the global minimum variances. Given that the frontiers are calculated using a time series of 75 observations, the lower bound on the test-statistic is

$$\xi_W^{span} \geq T \left( \frac{(\hat{\sigma}_R^0)^2}{(\hat{\sigma}^0)^2} - 1 \right) = 75 \left( \frac{0.13^2}{0.07^2} - 1 \right) = 183.7.$$

The  $p$ -value associated with a  $\chi^2$  with 36 degrees of freedom (18 emerging markets) is 0.000, implying that the hypothesis of spanning can easily be rejected. Harvey (1995) similarly rejects the null hypothesis that there is intersection at some point for the two frontiers at any conventional significance level. Therefore, unlike the developed markets, the emerging markets appear to offer diversification benefits to US-investors that are economically and statistically significant.

These results are further corroborated by testing a one-factor model where the cross-section of expected returns on the emerging markets is explained by their covariation with the world portfolio. Specifically, Harvey tests whether the intercepts in the regressions

$$r_{i,t+1} - R_t^f = \alpha_i + \beta_i (R_{t+1}^{world} - R_t^f) + \varepsilon_{i,t+1} \quad (82)$$

are equal to zero for all  $i$ . Here  $r_{i,t+1}$  is the return on emerging market  $i$ ,  $R_{t+1}^{world}$  is the return on the MSCI World Index, and  $R_t^f$  is the return on a 30-days Eurodollar deposit. Harvey motivates the use of (82) as a test of the World CAPM, which implies that  $\alpha_i = 0$  for each emerging market  $i$ . From the results in Section 5 it is clear that  $\alpha_i$  is the Jensen measure for emerging market  $i$  relative to the world portfolio. The test whether or not  $\alpha_i = 0$  is also a test whether there is intersection with the world portfolio as the benchmark asset and the zero beta rate equal to the risk free rate. Thus, instead of motivating (82) by the World CAPM and testing whether stock returns in emerging markets can be explained by their covariation with the world portfolio, (82) can also be motivated by the question whether an investor that initially holds the MSCI World Index (plus the risk free asset) can improve his efficient set by additional investments in emerging markets.

For the individual emerging markets, the estimated annualized intercepts are in the range between -16.65% (Indonesia) and 63.42% (Argentina) and 5 out of 20 intercepts are significantly different from zero (Argentina, Chile, Colombia, Pakistan and Philippines). Similar results are reported for a sample period that ends in June 1996 in DeRoos, Nijman & Werker (1998a). The regression estimates as reported by Harvey (1995) can be used to obtain information about the attainable Sharpe ratios and the new optimal portfolio weights for the assets considered. This will be illustrated in detail in the next Section for the Fama & French three-factor model. A joint test whether the intercepts of all 18 emerging markets are zero is rejected at any conventional significance level (the  $p$ -value is 0.001). Thus the hypothesis that the MSCI World Index spans the 18 emerging markets is convincingly rejected, implying that there are significant diversification benefits to an investor that initially only holds the world portfolio and that the World CAPM can not explain the cross section of emerging market returns.

Harvey (1995) also tests whether the intercepts in a two-factor model are equal to zero, i.e., whether in the regression

$$r_{i,t+1} - R_t^f = \alpha_i + \beta_{1,i}(R_{t+1}^{world} - R_t^f) + \beta_{2,i}(R_{t+1}^{FX} - R_t^f) + \varepsilon_{i,t+1}, \quad (83)$$

$\alpha_i = 0, \forall i$ . Here  $R_{t+1}^{FX}$  is the return on a trade-weighted portfolio of Eurocurrency deposits in 10 countries. The regression in (83) is motivated by the international asset pricing model of Adler & Dumas (1983). The estimated (annualized) intercepts for this model are in the range between -12.97% (Indonesia) and 64.06% (Argentina). Again, 5 out of the 20 intercepts are significantly different from zero (Argentina, Chile, Colombia, Philippines and Taiwan) and the  $p$ -value associated with a test for the hypothesis that the intercepts of all 18 emerging markets are zero is 0.001. Thus, the model of Adler & Dumas can not explain the cross section of emerging market returns either and US-investors that initially hold the world portfolio plus a trade-weighted portfolio of Eurocurrency deposits can extend their efficient set significantly by investing in the emerging markets.

Similar conclusions about the diversification benefits of emerging markets are reported by DeSantis (1994) for instance. Also, Errunza, Hogan & Hung (1998) study portfolios constructed from US-traded securities that mimic emerging markets indices, but their spanning tests show that direct investments in the emerging markets yield diversification benefits beyond the US-traded securities. As noted by Bekaert & Urias (1996) however, a drawback



of many studies on the diversification benefits of emerging markets is that the IFC Global Indices that are used in studies on emerging markets ignore the high transaction costs, low liquidity and investment constraints associated with emerging markets. Therefore, the diversification benefits suggested by these studies may not be attainable in real life. Bekaert & Urias try to overcome this problem by using the returns on emerging market closed-end country funds. Since these country funds are traded in the US-market itself for instance, they provide an indirect investment opportunity in emerging markets that is attainable to US-investors. Based on emerging market country funds in the US and the UK, Bekaert & Urias (1996) find only mixed evidence for the diversification benefits of emerging markets. This suggests that market frictions such as transaction costs and short sales constraints in the emerging markets may indeed prevent investors from realizing the diversification benefits of emerging markets. DeRoos, Nijman & Werker (1998a) study the effect of short sales constraints and transaction costs on tests for spanning and intersection in more detail and analyze the consequences of such market frictions for direct investments in emerging markets.

### 7.3 The Fama-French three-factor asset pricing model

In a recent paper Fama & French (1996) propose a three-factor model to explain cross-sectional variations in asset returns. It is well known that the static CAPM can not explain many patterns in stock returns that are related to size, book-to-market equity (BE/ME), cash flow/price (C/P), earnings/price (E/P), and past sales growth. Also, stocks with low returns in the long-term (five year) past appear to have high expected future returns and stocks that have had a high return in last year also have high expected future returns (momentum), findings that can not be explained by the static CAPM.

To illustrate these kinds of effects, Fama & French (1996) sort the NYSE, AMEX and Nasdaq stocks based on, e.g., their E/P ratio at the end of June of each year. These stocks are then allocated to ten portfolios, based on the decile breakpoints for E/P ratio's. For each of these ten portfolios monthly returns (equal weighted or value weighted) are calculated from July until the next June. This procedure is repeated for each year from July 1963 until December 1993. In a similar way, portfolios are formed based on BE/ME deciles, C/P deciles etc. For some variables also double sort portfolios are constructed. For instance, when sorting on BE/ME and (past) Sales, Fama & French sort the stocks independently on the basis of three BE/ME groups



and three Sales groups, resulting in a total of 9 portfolios.

Denote the return on a portfolio as  $r_{i,t+1}$ . Given a risk free rate  $R_t^f$  and the return on the market portfolio,  $R_{t+1}^m$ , the CAPM implies that in the regression

$$r_{i,t+1} - R_t^f = \alpha_i + \beta_i(R_{t+1}^m - R_t^f) + \varepsilon_{i,t+1} \quad (84)$$

$\alpha_i = 0, \forall i$ . Notice that  $\alpha_i$  is simply the classical Jensen measure. In other words, according to the CAPM the market portfolio and the risk free asset span all assets or portfolios  $i$ , as outlined in Section 3. Thus, with a risk free asset available (which in Fama & French (1996) is the one-month T-Bill rate), a test for the validity of the CAPM is simply a test whether the market portfolio intersects (spans) all other assets or portfolios in the economy. For each set of portfolios (i.e., based on a particular sort), Table 2 presents the average absolute intercept of the regression in (84) as well as the Gibbons-Ross-Shanken (GRS) test for zero-intercepts in (84). As noted in Section 5.3, the GRS-test is the small-sample version of the test in (66).

Table 2: Summary of intercepts and of spanning tests based on the CAPM. The results in the table are taken from Table IX in Fama & French (1996). Average absolute intercepts, intersection tests, changes in Sharpe ratios, and specification error bounds are shown when sorted portfolios are added to the market portfolio. Portfolios are sorted on size and book-to-market equity (double sort), earnings-price, past sales growth, cash flow/price and past sales growth (double sort), long-term past returns, which are from 60-13 months before formation, and short-term past returns, which are from 12-2 months before formation.

Portfolio	Avg. ( $ \alpha_i $ )	GRS	$p(\text{GRS})$	$\Delta \text{Sharpe}$	$\hat{\delta}/v$
<i>Size&amp;BE/ME</i>	0.286	2.76	(0.000)	0.362	0.453
<i>E/P</i>	0.260	2.85	(0.002)	0.201	0.285
<i>Sales</i>	0.256	2.51	(0.006)	0.184	0.267
<i>C/P&amp;Sales</i>	0.268	2.93	(0.002)	0.190	0.274
<i>returns(-60,-13)</i>	0.268	2.51	(0.006)	0.184	0.267
<i>returns(-12,-2)</i>	0.337	5.13	(0.000)	0.293	0.382

The 10 E/P sorted portfolios produce an average (absolute) intercept of 0.260. The GRS-test, which is calculated as

$$GRS = \frac{T - N - K}{N} \frac{\hat{\theta}(R_t^f)^2 - \hat{\theta}_R(R_t^f)^2}{1 + \hat{\theta}_R(R_t^f)^2}, \quad (85)$$

and which is (under the null-hypothesis of spanning and if all asset returns are jointly normally distributed and *i.i.d.*)  $F_{T-N-K,N}$ -distributed, is equal to 2.85 for the E/P-based portfolios. The summary statistics in Fama & French (1996) imply that the Sharpe ratio of the market portfolio is  $\hat{\sigma}_R(R_t^f) = 0.102$ . Since there is only one benchmark asset (the market portfolio), we have  $K = 1$ , and for the E/P-portfolios we have  $N = 10$ . Given that the sample size is 366, it follows that investing in the ten E/P-based portfolios besides the market portfolio, causes an increase in the maximum attainable Sharpe ratio from 0.102 to 0.302, which is not only significant in statistical terms, but also in economic terms. Similar results are also reported for the other sorts in Table 2.

The results in Table 2 illustrate some well-known CAPM-anomalies: empirical regularities in stock returns that can not be explained by the static CAPM. Fama & French (1996) claim that most of these anomalies are captured by their three-factor model, which states that the expected excess return on portfolio  $i$  is

$$E[r_{i,t+1} - R_t^f] = \beta_i^m E[R_{t+1}^m - R_t^f] + \beta_i^s E[R_{t+1}^{SMB}] + \beta_i^h E[R_{t+1}^{HML}], \quad (86)$$

where  $R_{t+1}^m$  and  $R_t^f$  are defined as before,  $R_{t+1}^{SMB}$  is the difference between a return on a portfolio of small stocks and the return on a portfolio of big stocks, and where  $R_{t+1}^{HML}$  is the difference between the return on a portfolio of high book-to-market stocks and the return on a portfolio of low book-to-market stocks. The small, big, high book-to-market and low book-to-market portfolios are created in a similar way as the portfolios described above. According to (86) expected stock returns are not only explained by the covariance of stock returns with the market ( $\beta_i^m$ ) as the CAPM predicts, but also by their covariation with  $R_{t+1}^{SMB}$  and  $R_{t+1}^{HML}$ . The loading on  $E[R_{t+1}^{SMB}]$ ,  $\beta_i^s$ , captures the well known size-effect. Small firms have average returns that can not be captured by the market return (see, e.g., Huberman & Kandel (1987)). Fama & French (1996) interpret  $\beta_i^h E[R_{t+1}^{HML}]$  as a premium for relative distress of a firm. They claim that weak firms tend to have high BE/ME ratios and positive slope coefficients  $\beta_i^h$ . Because  $E[R_{t+1}^{HML}] > 0$ , firms in distress will have higher expected returns.

Notice that  $R_{t+1}^{SMB}$  and  $R_{t+1}^{HML}$  are zero-investment positions. However, if these positions are combined with an investment in the risk free asset, then portfolios are created with return  $R_{t+1}^{SMB} + R_t^f$  and  $R_{t+1}^{HML} + R_t^f$  respectively. We will refer to these portfolios as the *SMB*-portfolio and the *HML*-portfolio.

$R_{t+1}^{SMB}$  and  $R_{t+1}^{HML}$  can therefore be interpreted as the excess returns on these portfolios. From Section 3 it follows that Equation (86) implies that the mean-variance frontier of the market portfolio and the *SMB* and *HML*-portfolios intersects (spans) the mean-variance frontier of these same portfolios plus all other portfolios, for a known risk free rate  $R_t^f$ . Therefore, the model in (86) can be tested by testing whether in the regression

$$r_{i,t+1} - R_t^f = \alpha_i + \beta_i^m(R_{t+1}^m - R_t^f) + \beta_i^s R_{t+1}^{SMB} + \beta_i^h R_{t+1}^{HML} + \varepsilon_{i,t+1}, \quad (87)$$

$\alpha_i = 0, \forall i$ . Notice again that  $\alpha_i$  is the generalized Jensen measure for the Fama & French three-factor model. Because Fama & French (1996) also present the GRS-tests based on (87), we can construct a table similar to Table 2.6 for the three-factor model as well. From the summary statistics the maximum Sharpe ratio for the market portfolio and the *SMB* and *HML*-portfolios can be calculated, which is 0.261. Since there are now three benchmark portfolios,  $K = 3$ . Using the reported GRS test-statistics in Fama & French (1996) it is then straightforward to calculate the increase in the maximum Sharpe ratios that may result from adding the portfolios based on the sorts in Table 2. These calculations are reported in Table 3.

The increases in the Sharpe ratios in Table 3 are much smaller than in Table 2. For instance, adding portfolios based on an E/P-sort to the market portfolio only, yields an increase in the Sharpe ratio of 0.201 in Table 2. Starting from the market portfolio and the *SMB* and *HML*-portfolios, adding the E/P-based portfolios yields an increase in the Sharpe ratio of only 0.045. Also, as the GRS-test shows, this latter increase is not statistically significant.

From the discussion in Section 6.2, recall that the specification error bound introduced by Hansen & Jagannathan (1997) is a function of the Sharpe ratios and the expectation of the kernel,  $v$ :  $\delta = v(\theta(1/v)^2 - \theta_R(1/v)^2)^{\frac{1}{2}}$ . The value of  $\delta/v$  is reported in the last columns of Table 2 and 3. Notice that  $v$  is not reported by Fama & French (1996), but  $v$  will be close to one, so  $\hat{\delta} \approx \delta/v$ . Except for the last rows of Table 2 and 3, which will be discussed in detail below, notice that the specification error bounds are much smaller in Table 3 than in Table 2. The reported bounds in Table 2 are mostly of the same size as the bounds reported for the market model by Hansen & Jagannathan (1997), which are approximately 0.29. Thus, the specification error bounds confirm that the three-factor model shows less misspecification than the CAPM, although the bounds in Table 3 are still rather large. For

Table 3: Summary of intercepts and of spanning tests based on the Fama-French three-factor model.

The results in the table are taken from Table IX in Fama & French (1996). Average absolute intercepts, intersection tests, changes in Sharpe ratios, and specification error bounds are shown when sorted portfolios are added to the three factor portfolios of Fama & French (1996) (market, SMB, and HML). Portfolios are sorted on Size and book-to-market equity (double sort), earnings-price, past sales growth, cash flow/price and past sales growth (double sort), long-term past returns, which are from 60-13 months before formation, and short-term past returns, which are from 12-2 months before formation.

Portfolio	Avg. ( $ \alpha_i $ )	GRS	$p(\text{GRS})$	$\Delta \text{Sharpe}$	$\hat{\delta}/v$
<i>Size&amp;BE/ME</i>	0.093	1.97	(0.004)	0.212	0.395
<i>E/P</i>	0.051	0.84	(0.592)	0.045	0.159
<i>Sales</i>	0.053	0.87	(0.563)	0.046	0.162
<i>C/P&amp;Sales</i>	0.062	1.04	(0.405)	0.049	0.168
<i>returns(-60,-13)</i>	0.092	1.29	(0.235)	0.066	0.198
<i>returns(-12,-2)</i>	0.331	4.46	(0.000)	0.190	0.367

instance, the bound derived from the P/E-based portfolios in table is 0.159, implying that in constructing portfolios from the three benchmark assets and the ten P/E-based portfolios, the three-factor model may imply a Sharpe ratio that is as far off as 0.159. Unfortunately, the results in Fama & French (1996) do not allow us to make an estimate of the standard error associated with this bound.

Although the results in Table 2 and 3 show that the three-factor model is much better able to explain expected stock returns than the static CAPM, there is still some evidence left against the three-factor model. First, the double-sorted portfolios on Size and BE/ME give an increase in the Sharpe ratio of 0.212 that is both economically and statistically significant, as the first row of Table 3 shows. The double sort on Size and BE/ME in Fama & French (1996) results in 25 portfolios. A closer look at the results in Fama & French (1996) shows that the three-factor model can explain most of the variation in portfolio returns, except for the smallest size stocks with the lowest BE/ME ratios, which have a large negative  $\hat{\alpha}_i$ , and the largest size stocks with the lowest BE/ME ratios, which have a large positive  $\hat{\alpha}_i$ . For the other portfolios the estimated  $\hat{\alpha}_i$  is close to zero. The main failure of



the three-factor model is in explaining returns for portfolios based on short-term past returns, as the last row of Table 3 shows. The portfolios labelled (-12,-2) are sorted on the return in the period 2-12 months prior to portfolio formation. These portfolios are meant to capture *momentum strategies* or *continuation* of short term returns. As shown in Table 3, investing in these portfolios besides investment in the three benchmark portfolios gives a significant improvement of the efficient set. Also, the results in Fama & French (1996) show that this improvement is almost uniform over the ten portfolios that are formed on the basis of returns in the period (-12,-2).

Finally, the results in Fama & French (1996) can be used to infer mean-variance efficient portfolios. The three-factor model suggests that (mean-variance) investors only have to invest in the market, the *SMB* and *HML*-portfolios and the risk free asset. Given the summary statistics in Fama & French (1996), the portfolio weights in the tangency portfolio can easily be calculated using standard mean-variance analysis (see, e.g., Ingersoll (1987), p.88-89). These weights are shown in the first column of Table 4. The expected excess return on the tangency portfolio is equal to 0.43% per month and the standard deviation of the portfolio return is 1.65%. As noted above,

Table 4: Portfolio weights for tangency portfolios of the Fama / French factor portfolios and of two short-term continuation portfolios.

	3-factor	+ cont(1)	+ cont(10)
<i>M</i>	0.25	0.35	-0.29
<i>SMB</i>	0.15	0.34	-0.17
<i>HML</i>	0.61	0.38	0.31
<i>cont(1)</i>		-0.21	
<i>cont(10)</i>			0.37

the short-term continuation portfolios are formed each year based on returns in the period 12-2 months before formation. Denote the return on the portfolio of stocks in the decile with the lowest returns as  $r_{t+1}^{(1)}$  and the return on the portfolio of stocks in the decile with the highest return as  $r_{t+1}^{(10)}$ . The estimated regressions for these portfolios as reported in Fama & French (1996) are:

$$\begin{aligned}
 r_{t+1}^{(1)} - R_t^f &= -1.15 + 1.14(R_{t+1}^m - R_t^f) + 1.35R_{t+1}^{SMB} + 0.54R_{t+1}^{HML} + \varepsilon_{t+1}^{(1)}, \\
 r_{t+1}^{(10)} - R_t^f &= 0.59 + 1.13(R_{t+1}^m - R_t^f) + 0.68R_{t+1}^{SMB} + 0.04R_{t+1}^{HML} + \varepsilon_{t+1}^{(10)}.
 \end{aligned}$$

Recall that the intercepts are the generalized Jensen measures, and that the sign of the intercept determines whether a mean-variance investor can improve the Sharpe ratio of his portfolio by taking a long or a short position. Thus, besides investing in the market and the *SMB* and the *HML*-portfolios, the investor can extend his efficient set by going short in the lowest decile portfolio (with return  $r_{t+1}^{(1)}$ ) or by buying the highest decile portfolio (with return  $r_{t+1}^{(10)}$ ).

Keeping the expected excess portfolio return constant at 0.43%, the new optimal portfolio weights can be determined using (62) and (63). The required estimate of  $\sigma(\varepsilon)$  that is needed for these weights can be derived from the Sharpe ratio of the three benchmark portfolios (0.261) and the *t*-values of the intercept, which for the lowest decile portfolio is -5.34 and for the highest decile portfolio 4.56. Using (65) it follows that  $\sigma(\varepsilon)$  is 3.99% and 2.32% for the lowest and the highest decile portfolios respectively. Given these estimates, the optimal portfolio weight for the lowest decile portfolio is equal to

$$w_r^{(1)} = \frac{0.43\%}{(0.261^2 + (-1.15\%/3.99\%)^2)} \frac{-1.15\%}{3.99\%} = -0.21.$$

Thus, to obtain the new maximum Sharpe ratio and have an expected excess portfolio return of 0.43%, the investor will need to take a short position in the lowest decile portfolio of 0.21. The funds obtained from selling this portfolio short are invested in the market portfolio and the *SMB*-portfolio, while he will also sell part of his holdings in the *HML*-portfolio, as can be seen in the second column of Table 4. Similar results are also reported for the highest decile portfolio in the third column of Table 4. Notice that both the benchmark portfolio and the continuation-based portfolios may contain any of the available stocks. Therefore, it is not clear from the reported results whether or not short positions in the individual stocks are required to realize the increase in the Sharpe ratios.

Summarizing, the results in Fama & French (1996) show that investors that initially hold the market portfolio of US stocks, can significantly improve their portfolio performance by using strategies based on well documented CAPM-anomalies, such as E/P, BE/ME, winner/loser and momentum strategies. This is not the case for investors that base their portfolio on the Fama & French three-factor model however. Investors that initially choose their mean-variance efficient portfolio from the market, the *SMB* and

the *HML*-portfolios can not reject the efficiency of their portfolio with respect to most of the strategies mentioned. The main exception appears to be caused by momentum strategies: Investing in portfolios that are formed on the basis of short-term past performance causes a shift in the mean-variance frontier of the three benchmark portfolios that is significant in both economic and statistical terms.

## 8 Summary and Concluding remarks

The purpose of this paper is to analyze and illustrate the concept of mean-variance spanning and intersection. We show that there is a duality between mean-variance frontiers and volatility bounds and that mean-variance spanning and intersection can be understood both in terms of mean-variance frontiers and volatility bounds. The paper shows how regression based tests can be used to test for spanning and intersection and how these regression based tests are related to tests for mean-variance efficiency, performance measurement, optimal portfolio choice and specification error bounds.

The framework developed in the paper is illustrated with some well studied empirical issues. We interpret some implications of studies on international diversification and currency hedging, and show that although for a US investor investing in other developed countries and adding currency forward contracts can result in Sharpe ratios that are much higher than the Sharpe ratio of US stocks only, the observed increases in Sharpe ratio's are usually statistically insignificant. Thus, the empirical evidence on international diversification appears to be rather weak. Diversification with investments in emerging markets on the other hand leads to diversification benefits that are both economically and statistically significant. Finally, we interpret the evidence on the Fama & French three-factor model in terms of Sharpe ratios and specification error bounds, and show that the Sharpe ratio from the three factor portfolios can be significantly improved upon by including investments in momentum strategies.

## A The graphical relationship between mean-variance frontiers and volatility bounds

In this appendix we will show some graphical relations between the volatility bound and the mean-variance frontier for a set of asset returns  $R_{t+1}$  with expectation  $\mu$  and covariance matrix  $\Sigma$ . We will start from a point on the volatility bound where the expectation of the minimum variance pricing kernel is  $v$ , i.e.,

$$E[m(v)_{t+1}] = v. \quad (88)$$

Using the efficient set variables  $A$ ,  $B$ , and  $C$ , and the variance of  $m(v)_{t+1}$  as given in (7), the variance of  $m(v)_{t+1}$  can be written as

$$\text{Var}[m(v)_{t+1}] = A - 2Bv + Cv^2, \quad (89)$$

which is a simple quadratic function of  $v$  that describes the volatility bound. Figure 1 gives a plot of  $\text{Var}[m(v)_{t+1}]$  as a function of  $v$ .

As shown in Section 2.2, each minimum variance pricing kernel  $m(v)_{t+1}$  corresponds to a mean-variance efficient portfolio that has a zero-beta rate  $\eta = 1/v$ . Recall that a mean-variance satisfies

$$w = \gamma^{-1} \Sigma^{-1} (\mu - \eta \iota),$$

for a given risk aversion  $\gamma$  and associated zero-beta rate  $\eta$ . Using  $\iota'w = 1$  it follows that

$$\gamma = B - \eta A.$$

Furthermore, the expected portfolio return  $\mu'w$  satisfies

$$\mu'w = \gamma^{-1} (C - \eta B) = \frac{C - \eta B}{B - \eta A}.$$

Denote the return on the mean-variance efficient portfolio with zero-beta rate  $\eta = 1/v$  as  $R(v)_{t+1}$  and define  $\mu(v) \equiv E[R(v)_{t+1}]$ . From the previous relations  $\mu(v)$  can be written as a function of  $v$ :

$$\mu(v) = \frac{B - Cv}{A - Bv}. \quad (90)$$

Also, the variance  $w'\Sigma w$  for a mean-variance efficient portfolio  $w$  can be written as a function of  $\mu(v)$ :

$$\text{Var}[R(v)_{t+1}] = \frac{A\mu(v)^2 - 2B\mu(v) + C}{AC - B^2},$$



or as a function of  $v$ :

$$\text{Var}[R(v)_{t+1}] = \frac{A - 2Bv + Cv^2}{(A - Bv)^2}. \quad (91)$$

Figure 2 shows the standard mean-variance efficient frontier, where the expected portfolio return  $\mu(v)$  is plotted as a function of the standard deviation of the portfolio return  $\text{stdev}[R(v)_{t+1}] = \text{Var}[R(v)_{t+1}]^{\frac{1}{2}}$ .

In this appendix we will restrict ourselves to characterizing the relation between the volatility bound and the mean-variance frontier in terms of  $v$  and  $\mu(v)$ . Given the relations (89) to (91) above it is straightforward to derive the variances of the pricing kernel and the associated mean-variance efficient portfolio as well.

To see the relation between the two graphs, first of all notice that the expected portfolio return  $\mu(v)$  is decreasing in  $v$ , since from (90) we have that

$$\frac{\partial \mu(v)}{\partial v} = \frac{B^2 - AC}{(A - vB)^2} < 0,$$

and where the inequality follows from the fact that  $AC > B^2$ , by the Cauchy-Schwarz inequality (see also Ingersoll (1987, p.85)).

Next, from (90) it also follows that for  $v = 0$  we have that  $\mu(v) = B/A$ , which is the expected return on the Global Minimum Variance portfolio. Looking at the volatility of the pricing kernel we can of course also distinguish the Global Minimum Variance Pricing Kernel, the expectation of which can be found using (89):

$$\begin{aligned} 0 &= \frac{\partial \text{Var}[m(v)_{t+1}]}{\partial v} = -2B + 2Cv^* \\ \Leftrightarrow v^* &= B/C. \end{aligned}$$

The second derivative  $2C$  is always positive, which confirms that this is indeed a minimum. Using (90) again,  $v = B/C$  corresponds to  $\mu(v) = 0$ . Thus, when the expectation of the kernel is zero,  $v = 0$ , this corresponds to the Global Minimum Variance portfolio on the mean-variance frontier, whereas a zero expected return for the mean-variance efficient portfolio,  $\mu(v) = 0$ , in turn corresponds to the Global Minimum Variance kernel on the volatility bound.

Having characterized the global minima of the two frontiers, the next step is to look at the other extremes, i.e., where  $v \rightarrow \pm\infty$  and where  $\mu(v) \rightarrow \pm\infty$ .

Taking limits and using (90) we get that

$$\begin{aligned}\lim_{v \rightarrow -\infty} \frac{B - Cv}{A - Bv} &= \frac{C}{B}, \\ \lim_{v \rightarrow +\infty} \frac{B - Cv}{A - Bv} &= \frac{C}{B}.\end{aligned}$$

Thus, both extremes of the left and right limb of the volatility bound correspond to the same single point on the mean-variance frontier, where the expected portfolio return is  $\mu(v) = C/B$ . Since by the Cauchy-Schwarz inequality  $C/B > B/A$  if  $B > 0$ , the point where  $\mu(v) = C/B$  will plot on the upper limb of the mean-variance frontier.  $B > 0$  is the typical case, since this implies that with positive interest rates or zero-beta returns, efficient portfolios have positive expected returns. It is useful to note that  $\mu(v) = C/B$  corresponds to the point where a straight line through the origin is tangent to the mean-variance frontier (since  $v \rightarrow \pm\infty$  corresponds to  $\eta = 0$ ).

Finally, by rewriting (90) as

$$v = \frac{B - A\mu(v)}{C - B\mu(v)},$$

we can find the point(s) on the volatility bound that correspond to the extremes of the mean-variance frontier, i.e., where  $\mu(v) \rightarrow \pm\infty$ . Taking limits again, we get that

$$\begin{aligned}\lim_{\mu(v) \rightarrow -\infty} \frac{B - A\mu(v)}{C - B\mu(v)} &= \frac{A}{B}, \\ \lim_{\mu(v) \rightarrow +\infty} \frac{B - A\mu(v)}{C - B\mu(v)} &= \frac{A}{B}.\end{aligned}$$

Notice that we already discussed this result in Section 2 since  $v = A/B \Leftrightarrow \eta = B/A$ , i.e. the case where the zero-beta return equals the expected return on the Global Minimum Variance portfolio and where there are no corresponding mean-variance efficient portfolios, since the asymptotes of the mean-variance frontier cross the  $y$ -axis at  $B/A$ , but there is no line tangent to the frontier starting at this point. Again, if  $B > 0$ , then the Cauchy-Schwarz inequality implies that  $A/B > B/C$ , implying that this point will be located on the right limb of the volatility bound. Finally, it is useful to note that if we would plot the volatility bound as the *standard deviation* of the pricing kernel,  $\text{Var}[m(v)_{t+1}]^{\frac{1}{2}}$ , as a function of  $v$ , then  $v = A/B$  would correspond to the point where a straight line through the origin is tangent to the volatility bound, similar to the mean-variance frontier when  $\mu(v) = C/B$ .

## B Consistency of the OLS-estimates when incorporating conditioning variables

The purpose of this appendix is to prove the result referred to in Section 4.2. Let the asset returns be described by

$$\begin{aligned} R_{t+1} &= \gamma'_R Z_t + \varepsilon_{R,t+1}, \\ r_{t+1} &= \gamma'_r Z_t + \varepsilon_{r,t+1}, \end{aligned}$$

with  $E[\varepsilon_{R,t+1}] = E[\varepsilon_{r,t+1}] = 0$ ,  $E[\varepsilon_{R,t+1} \varepsilon_{r,t+1}] = 0$ , and  $\varepsilon_{R,t+1}$  and  $\varepsilon_{r,t+1}$  jointly *i.i.d.* with variances and covariances given by  $\Omega_{RR}$ ,  $\Omega_{rr}$ , and  $\Omega_{rR}$ , then in the regression

$$R_t^f = \gamma x_t + \delta R_{t+1} + u_{t+1},$$

with  $E[u_{t+1}] = 0$ ,  $E[u_{t+1} x_t] = 0$ , and  $E[u_{t+1} R_{t+1}] = 0$ , the OLS-estimates  $\hat{\gamma}$  and  $\hat{\delta}$  are given by

$$\hat{\gamma} = (\hat{\gamma}'_r - \hat{\gamma}'_R \hat{\Omega}_{RR}^{-1} \hat{\Omega}_{Rr}) \quad \text{and} \quad \hat{\delta} = \hat{\Omega}_{RR}^{-1} \hat{\Omega}_{Rr}.$$

To see this, first rewrite the regression model as

$$r = \begin{pmatrix} X & R \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} + u,$$

with  $r$  a  $T \times N$  matrix,  $X$  a  $T \times (L+1)$  matrix,  $R$  a  $T \times K$  matrix,  $\gamma$  a  $(L+1) \times N$  matrix,  $\delta$  a  $K \times N$  matrix, and  $u$  a  $T \times N$  matrix of error terms. Define the idempotent matrix  $M$  as

$$M = I_T - X(X'X)^{-1}X'.$$

The next thing to note is that  $\Omega_{rR}\Omega_{RR}^{-1}$  follows from a regression of  $\varepsilon_{r,t+1}$  on  $\varepsilon_{R,t+1}$ . Using that the OLS-residuals  $e_r$  and  $e_R$  are given by

$$\begin{aligned} e_r &= r - X(X'X)^{-1}X'r = M'r, \\ e_R &= R - X(X'X)^{-1}X'R = M'R, \end{aligned}$$

this immediately suggests that an estimate of  $\Omega_{rR}\Omega_{RR}^{-1}$  can be obtained from

$$(e'_r e_R)(e'_R e_R)^{-1} = (r' M M' R)(R' M M' R)^{-1} = (r' M R)(R' M R)^{-1}.$$

Using partitioned inverses, we can write for the inverse of  $(X \ R)'(X \ R)$ :

$$\begin{aligned} & \left( \begin{pmatrix} X & R \end{pmatrix}' \begin{pmatrix} X & R \end{pmatrix} \right) \\ &= \begin{pmatrix} (X'X)^{-1} + (X'X)^{-1}X'R(R'MR)^{-1}R'X(X'X)^{-1} & -(X'X)^{-1}X'R(R'MR)^{-1} \\ -(R'MR)^{-1}R'X(X'X)^{-1} & (R'MR)^{-1} \end{pmatrix}. \end{aligned}$$

The OLS estimates of  $\gamma$  and  $\delta$  can now be written as

$$\begin{aligned} \hat{\gamma} &= (X'X)^{-1}X'r + (X'X)^{-1}X'R(R'MR)^{-1}R'X(X'X)^{-1}X'r - (X'X)^{-1}X'R(R'MR)^{-1}R'r \\ &= \hat{\gamma}_r + \hat{\gamma}_R\{(R'MR)^{-1}R'X(X'X)^{-1}X'r - (R'MR)^{-1}R'r\} \\ &= \hat{\gamma}_r + \hat{\gamma}_R\{(R'MR)^{-1}R'Mr\} = (\hat{\gamma}_r - \hat{\gamma}_R'\hat{\Omega}_{RR}^{-1}\hat{\Omega}_{Rr}), \end{aligned}$$

and

$$\begin{aligned} \hat{\delta} &= (R'MR)^{-1}R'r - (R'MR)^{-1}R'X(X'X)^{-1}X'r \\ &= (R'MR)^{-1}\{R'r - R'X(X'X)^{-1}X'r\} \\ &= (R'MR)^{-1}R'Mr = \hat{\Omega}_{RR}^{-1}\hat{\Omega}_{Rr}, \end{aligned}$$

which is what we wanted to show.

## C The spanning test-statistic in terms of Sharpe ratios

In this appendix we show how the spanning test statistic can be interpreted in terms of Sharpe ratios, a result that was presented in Section 5.3. Recall from Section 5.3 that the covariance matrix of the OLS-estimates  $\hat{b}$  equals

$$\Sigma_{\varepsilon\varepsilon} \otimes T^{-1} \begin{pmatrix} 1 + \mu_R'\Sigma_{RR}^{-1}\mu_R & -\mu_R'\Sigma_{RR}^{-1} \\ -\Sigma_{RR}^{-1}\mu_R & \Sigma_{RR}^{-1} \end{pmatrix}.$$

Premultiplying with  $H_{span}$  and postmultiplying with  $H_{span}$  as defined in (54) yields

$$\begin{aligned} & H_{span} \left( \Sigma_{\varepsilon\varepsilon} \otimes T^{-1} \begin{pmatrix} 1 + \mu_R'\Sigma_{RR}^{-1}\mu_R & -\mu_R'\Sigma_{RR}^{-1} \\ -\Sigma_{RR}^{-1}\mu_R & \Sigma_{RR}^{-1} \end{pmatrix} \right) H_{span}' \\ &= \Sigma_{\varepsilon\varepsilon} \otimes T^{-1} \begin{pmatrix} 1 + C_R & -B_R \\ -B_R & A_R \end{pmatrix}, \end{aligned} \tag{92}$$



the inverse of which is

$$\Sigma_{\varepsilon\varepsilon}^{-1} \otimes \frac{T}{A_R(1+C_R) - B_R^2} \begin{pmatrix} A_R & B_R \\ B_R & 1+C_R \end{pmatrix}. \quad (93)$$

Similarly, for  $h_{span}$  in (54) we have

$$\left( I_N \otimes \begin{pmatrix} 1 & 0'_K \\ 0 & \iota'_K \end{pmatrix} \right) b - I_N \otimes \begin{pmatrix} 0 \\ \iota \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \iota_K - 1 \\ \alpha_2 \\ \beta_2 \iota_K - 1 \\ \vdots \\ \alpha_N \\ \beta_N \iota_K - 1 \end{pmatrix}. \quad (94)$$

Premultiplying (93) with  $h_{span}$  and postmultiplying with  $h'_{span}$ , we get, after replacing population moments by their sample equivalents:

$$\xi_W^{span} = T \frac{\hat{A}_R \hat{\alpha}' \hat{\Sigma}_{\varepsilon\varepsilon}^{-1} \hat{\alpha} - 2 \hat{B}_R \hat{\alpha}' \hat{\Sigma}_{\varepsilon\varepsilon}^{-1} (\iota_N - \hat{\beta} \iota_K) + (1 + \hat{C}_R) (\iota_N - \hat{\beta} \iota_K)' \hat{\Sigma}_{\varepsilon\varepsilon}^{-1} (\iota_N - \hat{\beta} \iota_K)}{\hat{A}_R(1 + \hat{C}_R) - \hat{B}_R^2}. \quad (95)$$

Next note that the maximum attainable Sharpe ratio from  $R_{t+1}$ , for  $\eta = B_R/A_R$ , is equal to

$$\theta_R \left( \frac{B_R}{A_R} \right)^2 = C_R - \frac{B_R^2}{A_R}.$$

For simplicity, write  $A = A_R + \Delta A$ ,  $B = B_R + \Delta B$ , and  $C = C_R + \Delta C$ , where the definitions of  $\Delta A$ ,  $\Delta B$ , and  $\Delta C$  follow from (53) and (??). Evaluating  $\theta(\eta)$  in this same value of  $\eta$ , we get

$$\begin{aligned} \theta \left( \frac{B_R}{A_R} \right)^2 &= C - 2B \frac{B_R}{A_R} + A \frac{B_R^2}{A_R^2} \\ &= C_R + \Delta C - 2(B_R + \Delta B) \frac{B_R}{A_R} + (A_R + \Delta A) \frac{B_R^2}{A_R^2} \\ &= \theta_R \left( \frac{B_R}{A_R} \right)^2 + \frac{1}{A_R} \left( A_R \Delta C - 2B_R \Delta B + \frac{B_R^2}{A_R} \Delta A \right) \end{aligned}$$

Dividing by  $(1 + C_R) - B_R^2/A_R = 1 + \theta_R \left( \frac{B_R}{A_R} \right)^2$  gives

$$\frac{\theta \left( \frac{B_R}{A_R} \right)^2 - \theta_R \left( \frac{B_R}{A_R} \right)^2}{1 + \theta_R \left( \frac{B_R}{A_R} \right)^2} = \frac{A_R \Delta C - 2B_R \Delta B + \frac{B_R^2}{A_R} \Delta A}{A_R(1 + C_R) - B_R^2}$$

$$\begin{aligned}
&= \frac{A_R \Delta C - 2B_R \Delta B + \left( C_R + 1 - 1 - \theta \left( \frac{B_R}{A_R} \right)^2 \right) \Delta A}{A_R(1 + C_R) - B_R^2} \\
&= \frac{A_R \Delta C - 2B_R \Delta B + (1 + C_R) \Delta A}{A_R(1 + C_R) - B_R^2} - \frac{\Delta A}{A_R}.
\end{aligned}$$

Replacing all population moments with their sample equivalents again and noting that  $1/A_R$  is the variance of the global minimum variance portfolio of  $R_{t+1}$ , i.e.,  $1/A_R = (\sigma_R^0)^2$ , and similarly,  $1/A = (\sigma^0)^2$ , we finally obtain

$$\begin{aligned}
\xi_W^{span} &= T \frac{\hat{\theta} \left( \frac{B_R}{A_R} \right)^2 - \hat{\theta}_R \left( \frac{B_R}{A_R} \right)^2}{1 + \hat{\theta}_R \left( \frac{B_R}{A_R} \right)^2} + T \frac{\hat{A} - \hat{A}_R}{\hat{A}_R} \\
&= T \left( \frac{1 + \hat{\theta}(\eta_R^0)^2}{1 + \hat{\theta}_R(\eta_R^0)^2} + \frac{(\hat{\sigma}_R^0)^2}{(\hat{\sigma}^0)^2} - 2 \right).
\end{aligned}$$

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# Volatility Bound

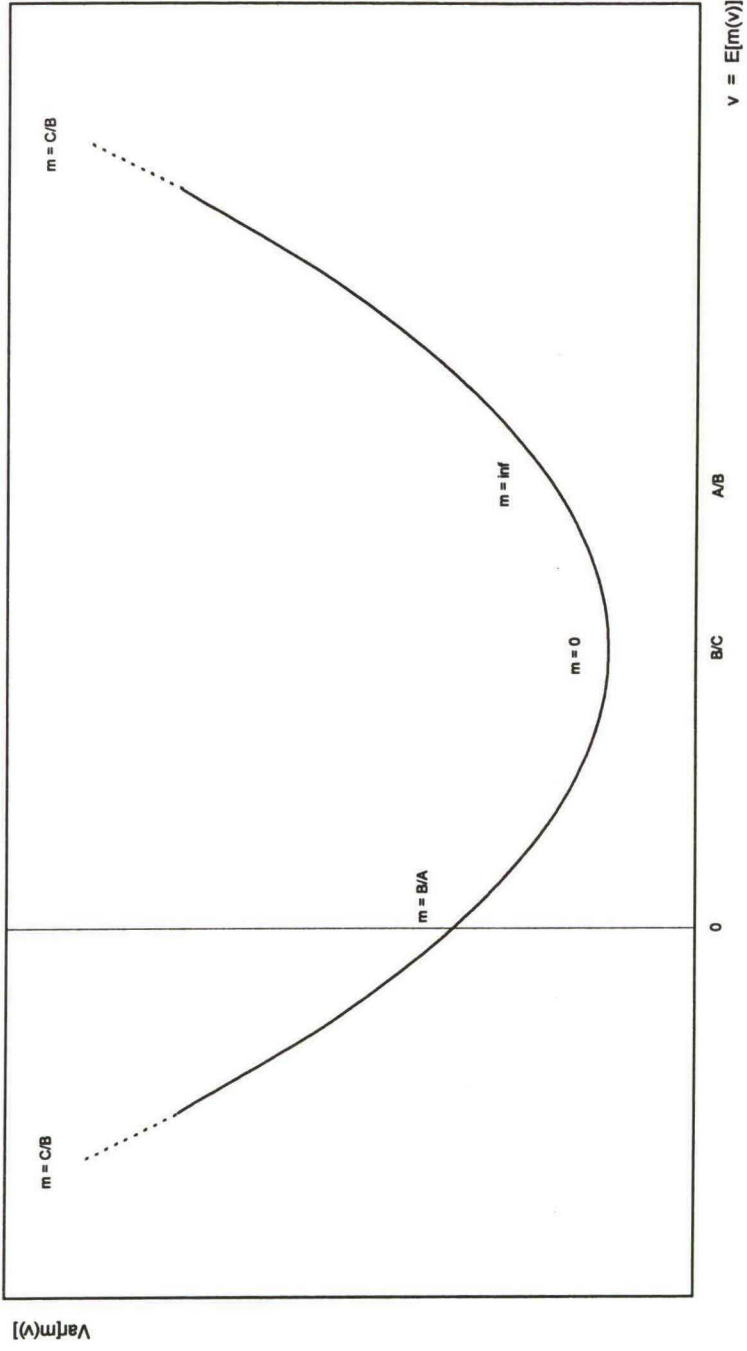


Figure 1

# Mean-Variance Frontier

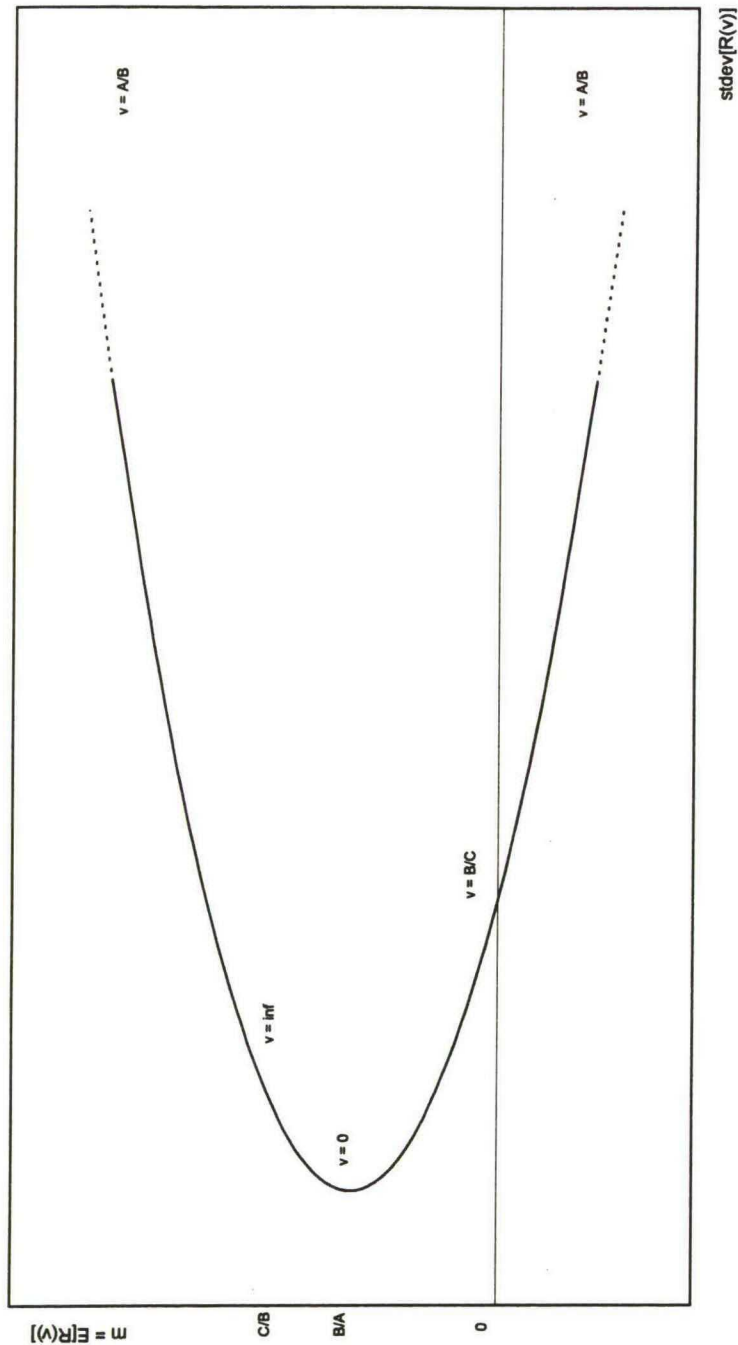


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